# Answers and Teachers' Notes


MINISTRY OF EDUCATION
*Te Tāhuhu o te Mātauranga*

## Contents

# Introduction

The books in the Figure It Out series are issued by the Ministry of Education to provide support material for use in New Zealand classrooms. In recent years, much of the Figure It Out student material has been aligned with Numeracy Development Project strategies, which are reflected in the answers and in the teachers' notes.

**Student books**

The activities in the student books are written for New Zealand students and are set in meaningful contexts, including real-life and imaginary scenarios. The level 3–4 contexts reflect the ethnic and cultural diversity and the life experiences that are meaningful to students in year 4. However, teachers should use their judgment as to whether to use the level 3–4 books with older or younger students who are also working at these levels.

The activities can be used as the focus for teacher-led lessons, for students working in groups, or for independent activities. Also, the activities can be used to fill knowledge gaps (hot spots), to reinforce knowledge that has just been taught, to help students develop mental strategies, or to provide further opportunities for students moving between strategy stages of the Number Framework.

***Answers and Teachers' Notes***

The Answers section of the *Answers and Teachers' Notes* that accompany each of the *Multiplicative Thinking* student books includes full answers and explanatory notes. Students can use this section for self-marking, or you can use it for teacher-directed marking. The teachers' notes for each activity, game, or investigation include relevant achievement objectives, Number Framework and other links, comments on mathematical ideas, processes, and principles, and suggestions on teaching approaches. The *Answers and Teachers' Notes* are also available on Te Kete Ipurangi (TKI) at www.tki.org.nz/r/maths/curriculum/figure

**Using Figure It Out in the classroom**

Where applicable, each page starts with a list of equipment that the students will need in order to do the activities. Encourage the students to be responsible for collecting the equipment they need and returning it at the end of the session.

Many of the activities suggest different ways of recording the solution to the problem. Encourage your students to write down as much as they can about how they did investigations or found solutions, including drawing diagrams. Discussion and oral presentation of answers is encouraged in many activities, and you may wish to ask the students to do this even where the suggested instruction is to write down the answer.

> The ability to communicate findings and explanations, and the ability to work satisfactorily in team projects, have also been highlighted as important outcomes for education. Mathematics education provides many opportunities for students to develop communication skills and to participate in collaborative problem-solving situations.
>
> *Mathematics in the New Zealand Curriculum,* page 7

Students will have various ways of solving problems or presenting the process they have used and the solution. You should acknowledge successful ways of solving questions or problems, and where more effective or efficient processes can be used, encourage the students to consider other ways of solving a particular problem.

Figure It Out

# Multiplicative Thinking Answers

## Page 1: RJ's Fabulous Juice Bar

### Activity

1. a. 120 mL. (300 ÷ 5 = 60. 60 x 2 = 120)

   b. A medium Pineapple Punch. Both cocktails require 2 parts of pineapple juice, but the pineapple juice in the Pineapple Punch is 2 parts of 3 (that is, $\frac{2}{3}$ of 600) and in the Hawaiian is 2 parts of 5 (that is, $\frac{2}{5}$ of 600).
   A medium Hawaiian needs
   600 ÷ 5 x 2 = 240 mL of pineapple juice.
   A medium Pineapple Punch needs
   600 ÷ 3 x 2 = 400 mL of pineapple juice.

   c. 250 mL. ($\frac{1}{4}$ of 1 L [1 000 mL])

2. 75 mL coconut milk (300 ÷ 4), 75 mL orange juice (300 ÷ 4), 150 mL mango juice (300 ÷ 4 x 2 or 300 ÷ 2)

3. Amounts will vary. For example, a tiny cocktail could be 150 mL or even 75 mL. A giant cocktail could be 2 or 3 times as big as a large cocktail. For both new sizes, the quantities need to be kept in proportion.

   a. Answers will vary. For example, for a 150 mL tiny cocktail, 1 part for a tiny Hawaiian would be 30 mL (150 ÷ 5), and the drink would be 30 mL orange, 60 mL pineapple, and 60 mL mango. A 75 mL tiny Hawaiian would be half of these quantities.

   b. Answers will vary. For a giant Tropical that was twice the large size, you would have 500 mL coconut milk, 1 L mango, and 500 mL orange; for 3 times the large size: 750 mL coconut milk, 1.5 L mango, and 750 mL orange.

4. Problems will vary. You may have chosen different mL for the size of your drink, along with different flavours and ingredients.

## Pages 2–3: Divided We Stand

### Activity

1. a. 72 ÷ 8 = 9, so 72 ÷ 4 = 18.

   b. 72 ÷ 8 = 9, so 72 ÷ 9 = 8.

   c. 8 x 9 = 72, so (using tripling and thirding) 24 x 3 = 72.

2. a. Some equations for 56 are:
   56 ÷ 8 = 7,
   so 56 ÷ 7 = 8,
   so 56 ÷ 14 = 4,
   so 56 ÷ 4 = 14,
   so 56 ÷ 2 = 28,
   so 56 ÷ 28 = 2.
   Or: 56 ÷ 1 = 56, so 56 ÷ 56 = 1.

   b. Some equations for 48 are:
   48 ÷ 6 = 8,
   so 48 ÷ 8 = 6,
   so 48 ÷ 4 = 12,
   so 48 ÷ 12 = 4,
   so 48 ÷ 24 = 2,
   so 48 ÷ 2 = 24;
   48 ÷ 3 = 16,
   so 48 ÷ 16 = 3.
   Or: 48 ÷ 1 = 48, so 48 ÷ 48 = 1.

   c. Some equations for 64 are:
   64 ÷ 8 = 8,
   so 64 ÷ 4 = 16,
   so 64 ÷ 2 = 32,
   so 64 ÷ 32 = 2,
   so 64 ÷ 16 = 4
   (or starting with their reverse equations, for example, 64 ÷ 16 = 4). Or: 64 ÷ 1 = 64, so 64 ÷ 64 = 1.

d. Some equations for 96 are:
96 ÷ 8 = 12,
so 96 ÷ 12 = 8,
so 96 ÷ 6 = 16,
so 96 ÷ 3 = 32;
96 ÷ 4 = 24,
so 96 ÷ 2 = 48
(or starting with their reverse equations, for example, 96 ÷ 48 = 2).

e. Some equations for 78 are:
78 ÷ 2 = 39,
so 78 ÷ 39 = 2;
78 ÷ 3 = 26,
so 78 ÷ 6 = 13,
so 78 ÷ 13 = 6.

f. Some equations for 84 are:
84 ÷ 4 = 21;
so 84 ÷ 12 = 7,
so 84 ÷ 6 = 14,
so 84 ÷ 3 = 28
(or their reverse equations).

3. a. Matthew's idea: Double the divisor gives half the answer. Examples will vary. An example of doubling the divisor is:
64 ÷ 8 = 8, so 64 ÷ 16 = 4.

b. Laurel's idea: The divisor and the answer are both factors of the first number. Examples will vary. One example (using the basic fact 5 x 9 = 45) is: 45 ÷ 5 = 9, so 45 ÷ 9 = 5.

c. Taylah's idea: Halving the number being divided and the divisor will give the same answer. Examples will vary. One example is: 42 ÷ 14 = 3, so 21 ÷ 7 = 3.

d. Quinten's idea: Treble the divisor, and the answer is $\frac{1}{3}$ of the first answer. Examples will vary. One example is: 36 ÷ 4 = 9, so 36 ÷ 12 = 3.

4. The choice of two strategies will vary, and so will the reasons given. Some examples are:

a. For 112 ÷ 8 = □, you could use:
56 ÷ 4 = 14 and 28 ÷ 2 = 14 (Taylah's strategy). Another strategy that works well here, halving the number being divided gives 2 times the answer, is a variation on Matthew's strategy: 56 ÷ 8 = 7, so 112 ÷ 8 = 14. To use this strategy in this problem, you need to know 8 x 7 = 56.

b. For 140 ÷ 5 = □, you could use:
140 ÷ 10 = 14, so 140 ÷ 5 = 28 (Matthew's strategy); or
280 ÷ 10 = 28 (reversing Taylah's strategy: doubling both instead of halving). Both work well as long as you know how to use your 10 times table in division.

c. For 81 ÷ 3 = □, you could use:
81 ÷ 9 = 9, so 81 ÷ 3 = 27 (from 9 x 3 = 27) (reversing Quinten's strategy); or
81 ÷ 6 = 13.5 and 13.5 x 2 = 27, so
81 ÷ 3 = 27 (Matthew's strategy). The first way is easier for this problem because it uses the 9 and 3 times tables and whole numbers.

d. For 88 ÷ 4 = □, you could use 44 ÷ 2 = 22, so 88 ÷ 4 = 22 (Taylah's strategy); or
88 ÷ 8 = 11, 11 x 2 = 22, so 88 ÷ 4 = 22 (Matthew's strategy). Both ways work well for this problem.

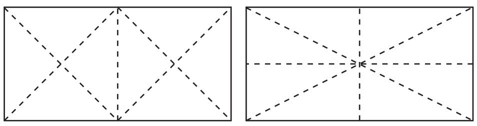
5. Problems will vary. They should all have an answer of 15. The equations used could include 30 ÷ 2 = 15, 120 ÷ 8 = 15, or 15 ÷ 1 = 15.

## Pages 4–5: Cutting It

### Activity

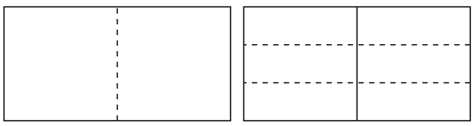
1. a. Mrs Cook halves, then halves, then halves again.

b. Drawings will vary. Two possible drawings are:



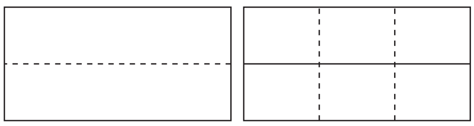
c. If Mrs Cook's first cut is a half, then drawings should be similar to:

i.



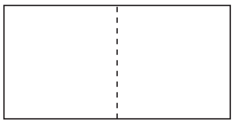
half thirds

or:

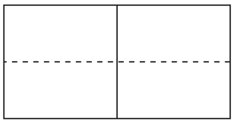


half thirds

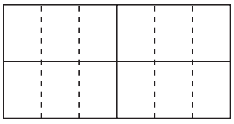
**ii.**



half

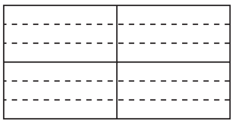


half



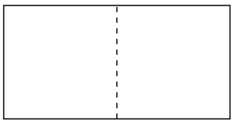
thirds

(or the cuts could be horizontal)

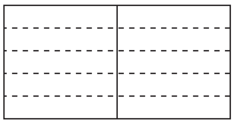


thirds

**iii.**

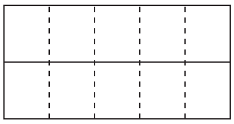


half



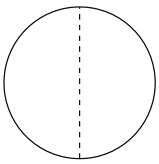
fifths

(or the cuts could be vertical)



fifths

**2.** **a.**



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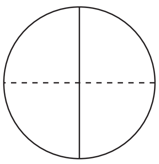
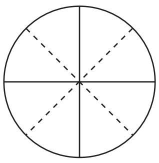


half



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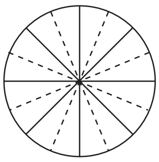
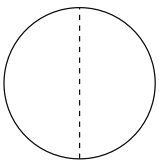


half

**b.**



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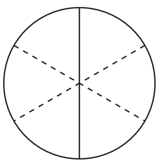
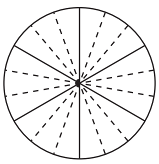
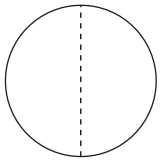


thirds

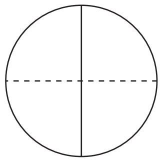


thirds

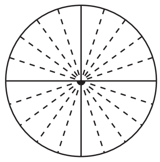
**c.**



half

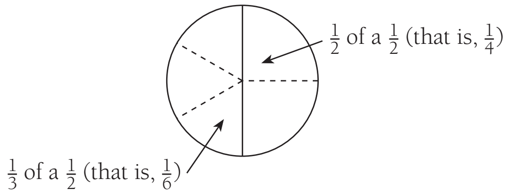


half

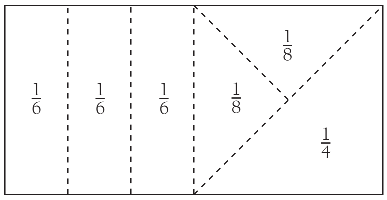


fifths

**3.** Annie is right. The diagram below shows why.

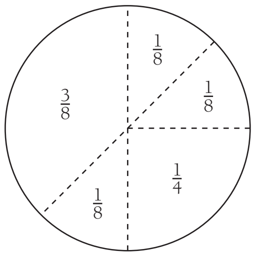



**4.** **a.** Explanations may vary.



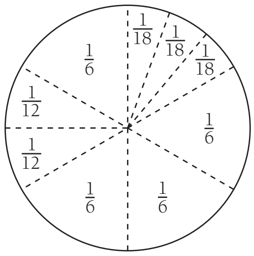

Half of the rectangle is divided into thirds. A third of $\frac{1}{2}$ is $\frac{1}{6}$. Half of the other $\frac{1}{2}$ is $\frac{1}{4}$, and $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$.

**b.** Explanations may vary.



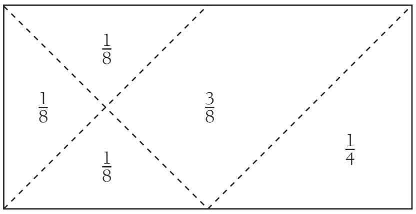

The right half is divided to give $\frac{1}{2}$ of a $\frac{1}{2}$ (that is, $\frac{1}{4}$) and 2 lots of $\frac{1}{2}$ of a $\frac{1}{4}$, which is 2 lots of $\frac{1}{8}$. The left half has one piece the same size as the $\frac{1}{8}$ on the other side, so the bigger segment must be $\frac{3}{8}$ because $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.

**c.** Explanations may vary.




The circle is divided into sixths and fractions of sixths. (If you look to the right of one of the centre lines, you can see $\frac{3}{6}$ in that half.) The top right parts are $\frac{1}{3}$ of $\frac{1}{6}$, that's $\frac{1}{18}$. The centre left parts are $\frac{1}{2}$ of $\frac{1}{6}$, that's $\frac{1}{12}$.

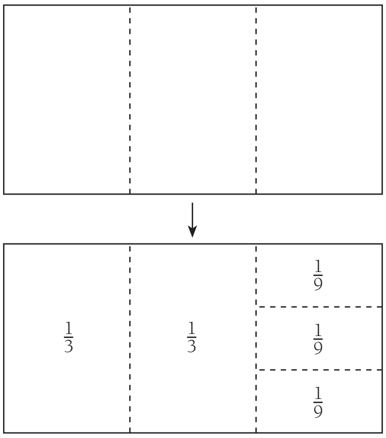
**d.** Explanations may vary.



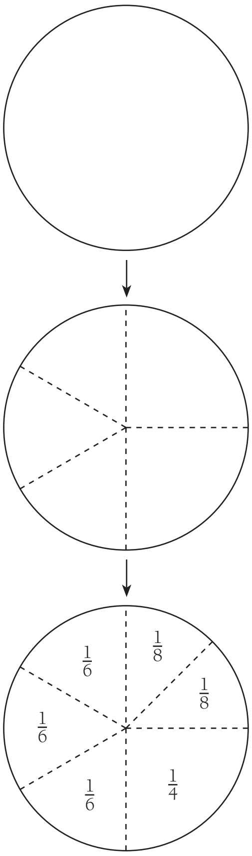

The bottom right-corner fraction is $\frac{1}{2}$ of a $\frac{1}{2}$, or $\frac{1}{4}$. From there, you can work out that there are three $\frac{1}{8}$ fractions, which means the last fraction must be $\frac{3}{8}$ ($\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$; $\frac{5}{8} + \frac{3}{8} = 1$).

**5.** Drawings will vary, depending on where you put your dividing lines. Possible drawings include:

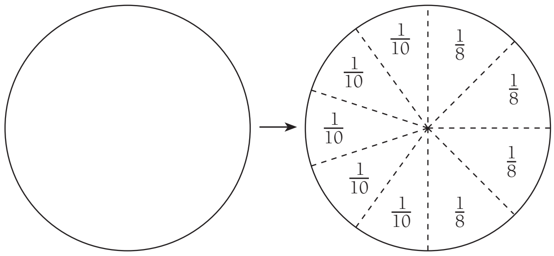
**a.**



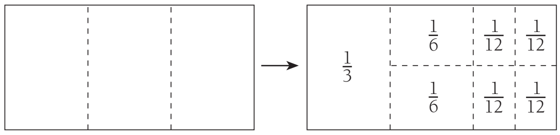

**b.**




**c.**




**d.**




**6.** Problems will vary.

## Pages 6–7: Fair Shares

### Activity

**1.** The completed equations are:
20 ÷ 4 = 5,
so 40 ÷ 8 = 5,
so 60 ÷ 12 = 5,
so 30 ÷ 6 = 5,
so 15 ÷ 3 = 5.
The pattern in these division equations is that doubling or halving both numbers produces the same answer. (60 ÷ 12 = 5, from 20 ÷ 4 = 5, is trebling.)

**2. a.** Grace used halving of both numbers to go directly from 72 ÷ 18 = 4 to 36 ÷ 9 = 4. Some other divisions using doubling and trebling patterns and basic multiplication facts are: 12 ÷ 3 = 4, 24 ÷ 6 = 4, 48 ÷ 12 = 4, and 144 ÷ 36 = 4.

**b.** They will all have the same answer because both numbers in each division statement are either doubles or halves of other division statements that have an answer of 4. Sharing half as many biscuits among half as many people gives the same share.

**3.** 168 ÷ 24 = 84 ÷ 12 = 42 ÷ 6 = 7 biscuits per person;
so 12 x 7 = 84 (at the 12-seat table)
and 6 x 7 = 42 (at the 6-seat table);
4 x 7 = 28 (at the 4-seat table),
and 3 x 7 = 21 (at the 3-seat table);
or 42 ÷ 6 = 7, so 84 ÷ 12 = 7, 21 ÷ 3 = 7,
and 28 ÷ 4 = 7.

4. a. 11 biscuits.
(396 ÷ 36 = 132 ÷ 12 = 66 ÷ 6 = 11)

b. 8 biscuits.
(224 ÷ 28 = 112 ÷ 14 = 56 ÷ 7 = 8)

c. 9 biscuits. (243 ÷ 27 = 81 ÷ 9 = 9)

5. Problems will vary.

## Pages 8–9: Leftovers

### Activity One

1. i. 10 ÷ 8 = 1.25

ii. 7 ÷ 8 = 0.875

iii. 13 ÷ 8 = 1.625

iv. 2.45 ÷ 8 = 0.30625

v. 5 x 30 ÷ 8 = 18.75

2. Of the calculator answers in question **1**, the useful ones are:

i. 10 ÷ 8 = 1.25 means 1.25 L or 1 250 mL.

ii. 7 ÷ 8 = 0.875 means 0.875 kg or 875 g.

iii. 13 ÷ 8 = 1.625 means 1.625 m or 162.5 cm or 1 625 mm.

3. a. Meaningless or not very sensible calculator answers:

iv. 2.45 ÷ 8 = 0.30625 is meaningless because 2.45 doesn't mean $2\frac{45}{100}$ hrs.

v. 5 x 30 ÷ 8 = 18.75 has some meaning because 18.75 means each group gets 18 biscuits; but 0.75 doesn't tell you how many biscuits will be left over. However, the calculator answer of 18.75 could be interpreted as $18\frac{3}{4}$ biscuits.

b. iv. 2 hrs and 45 min needs to be turned into min: (2 x 60) + 45 = 165 min. 165 ÷ 8 = 20.625 min. The 0.625 min can be ignored (for example, to allow for group changeovers), so each group gets 20 min on the flying fox.

v. 5 x 30 ÷ 8 = 18.75 means each group gets 18 whole biscuits. 0.75 x 8 = 6 means 6 biscuits are left over (or each group gets another 0.75 or $\frac{3}{4}$ of a biscuit!).

### Activity Two

1. a. i. 33 ÷ 5 = 6.6. So each group gets 6 whole punnets with 5 x 0.6 = 3 punnets left over. These 3 punnets could be shared in a variety of ways, for example, 4 groups could get $\frac{1}{2}$ a punnet while the fifth group gets an extra punnet.

ii. 24 ÷ 5 = 4.8 fish. So each group could get 4 whole fish. The remainder of 4 fish could be dealt with in several ways, for example, one group could get the 4 biggest fish while the other 4 groups could get 5 fish each.

iii. 57 ÷ 5 = 11.4 eggs. Each group gets 11 eggs with a remainder of 2 eggs. So 2 of the 5 groups could get 12 eggs.

b. Tessa is right. 0.6 is the same as $\frac{3}{5}$.
Chenda's answer of 6.3 means $6\frac{3}{10}$, and $\frac{3}{10}$ is not the same as $\frac{3}{5}$.

2. a. i. 19 ÷ 3 = $6.\dot{3}$ bags. So 2 groups could get 6 bags while the other group gets 7 bags, or they could share out the cherries in the extra bag.

ii. 21 ÷ 3 = 7 shellfish for each group

iii. 16 ÷ 3 = $5.\dot{3}$ buckets. So 2 groups get 5 buckets and the other group gets 6 buckets, or they could share out the mushrooms in the extra bucket.

b. Cherries: 19 ÷ 3 = $6\frac{1}{3}$ or $6.\dot{3}$
Shellfish: 21 ÷ 3 = 7 or 7.0
Mushrooms: 16 ÷ 3 = $5\frac{1}{3}$ or $5.\dot{3}$

3. Fractions are better if the remainders are whole numbers, but decimals are more useful if you are sharing out quantities measured in metric units.

## Pages 10–11: The Power of 10

### Activity

1. a. 12 ÷ 0.6 = 20. (Ways of using the diagram to show that 0.6 x 20 = 12 will vary. For example, you might work out that there are 120 units in the diagram; 120 ÷ 6 = 20. Or you might see 12 lots of 0.6, plus 2 lots of 0.6 in the top 4 units of each set of 3 columns [there are 4 sets of 3 columns]: that's 12 + (2 x 4) = 12 + 8 = 20.)

b. The answer to 12 ÷ 0.6 is 10 times bigger. (12 ÷ 0.6 = 20; 12 ÷ 6 = 2)

2. a. Equations **ii** and **iii**

   b. i. 10 times bigger.
      12 ÷ 6 = 2; 0.2 x 10 = 2

   ii. 100 times bigger.
      12 ÷ 0.6 = 20; 0.2 x 100 = 20

   iii. 1 000 times bigger. 12 ÷ 0.06 = 200; 0.2 x 1 000 = 200

   iv. 10 times smaller. 12 ÷ 600 = 0.02; 0.2 ÷ 10 = 0.02

3. a. 60 balloons. (18 ÷ 3 = 6, so 18 ÷ 0.3 = 60)

   b. 5 pkts of cheerios. (20 ÷ 4 = 5, so 2 ÷ 0.4 = 5)

   c. 25 kebab sticks. (700 ÷ 28 = 25, so 7 ÷ 0.28 = 25)

   d. 0.4 kg or 400 g. (60 ÷ 15 = 4, so 6 ÷ 15 = 0.4)

4. a. Answers will vary, but the divisor must be between 0 and 1, for example, 14 ÷ 0.7 = 20, 16 ÷ 0.5 = 32, 12 ÷ 0.8 = 15.

   b. Answers will vary, but the divisor must be 1, for example, 14 ÷ 1 = 14, 202 ÷ 1 = 202, 88 ÷ 1 = 88.

5. a. Multiply the answer to 32 ÷ 8 by 10 because 0.8 is $\frac{1}{10}$ of 8: 32 ÷ 8 = 4, so 32 ÷ 0.8 = 40.

   b. Divide the answer to 32 ÷ 8 by 10 because 80 is 10 times bigger than 8: 32 ÷ 8 = 4, so 32 ÷ 80 = 0.4.

   c. Divide the answer to 32 ÷ 8 by 100 because 800 is 100 times bigger than 8: 32 ÷ 8 = 4, so 32 ÷ 800 = 0.04.

   d. Multiply the answer to 32 ÷ 8 by 100 because 0.08 is $\frac{1}{100}$ of 8: 32 ÷ 8 = 4, so 32 ÷ 0.08 = 400.

   e. Divide the answer to 32 ÷ 8 by 10 because 3.2 is $\frac{1}{10}$ of 32: 32 ÷ 8 = 4, so 3.2 ÷ 8 = 0.4.

   f. Divide the answer to 32 ÷ 8 by 100 because 3.2 x 10 = 32 and 8 x 10 = 80: 32 ÷ 8 = 4, so 3.2 ÷ 80 = 0.04. You could also do this in two parts: 3.2 is $\frac{1}{10}$ of 32, so divide the answer to 32 ÷ 8 by 10: so 3.2 ÷ 8 = 0.4. 80 is 10 times bigger than 8, so divide the answer to 3.2 ÷ 8 by 10: so 3.2 ÷ 80 = 0.04.

## Pages 12–13: Spoilt for Choice

### Activity

1. a. First, you'd need to add extra boxes for the extra carbohydrate and fruit items. Then, each box where the carbohydrates and the fruit intersect could be divided into 4 to include the 4 drink options. There are 4 carbohydrates and 5 fruits, so there are 4 x 5 = 20 larger boxes, or 20 combinations using carbohydrates and fruit alone. If each of those options is then subdivided into 4 for the 4 drink options, there will be 20 x 4 = 80 different options. A diagram might look like this:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **filled roll** | juice | milk | juice | milk | juice | milk | juice | milk | juice | milk |
| | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup |
| **pita bread pocket** | juice | milk | juice | milk | juice | milk | juice | milk | juice | milk |
| | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup |
| **toasted sandwich** | juice | milk | juice | milk | juice | milk | juice | milk | juice | milk |
| | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup |
| **panini** | juice | milk | juice | milk | juice | milk | juice | milk | juice | milk |
| | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup | yoghurt | soup |
| | **apple** | | **orange** | | **banana** | | **kiwifruit** | | **plum** | |

b. Yes, the tuck shop's claim is true: there are 80 different options, and a school term has no more than 55 school days in an 11 week term.

2. 4 carbohydrate options x 5 fruit options x 4 drink options = 4 x 5 x 4 = 80 combinations

3. The extra item should be a carbohydrate or a drink. Adding an extra carbohydrate or drink item would give 100 different combinations (5 carbohydrates x 5 fruits x 4 drinks = 100 combinations or 4 carbohydrates x 5 fruits x 5 drinks = 100 combinations), while adding an extra fruit would only give 96 different options (4 carbohydrates x 6 fruits x 4 drinks = 96 combinations).

4. Answers will vary. For example, there could be 3 options for 3 of the categories and 2 for the fourth one: 3 x 3 x 3 x 2 = 54 different sandwich options. Other possibilities could include 2 x 5 x 5 x 1 = 50 or 2 x 4 x 4 x 2 = 64.

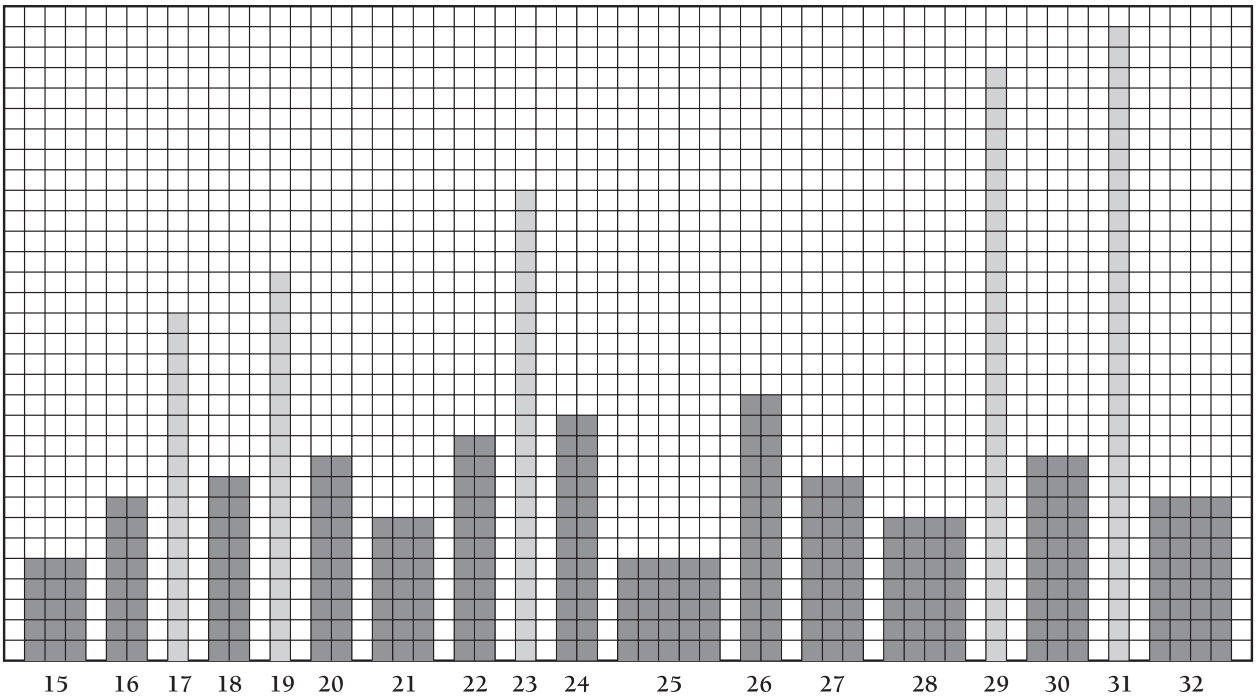
5. Discussion will vary. Allowing various combinations greatly increases the number of choices.

## Pages 14–15: Primes and Emirps

### Activity One

1. a. Yes, Hine is correct. The red numbers are prime numbers. Prime numbers only have two factors, themselves and 1, so they are always "thin" numbers. "Fat" numbers are not primes and have more than 2 factors. Two of their factors are themselves and 1. For example, 6 can be shown as a thin number because two of its factors are 6 and 1. But 2 and 3 are also factors of 6, so it is a fat number on the grid.

   b. Fat numbers may vary, depending on the factors you choose.




2. a. The only factor of 1 is 1.
The factors of 2 are 2, 1.
The factors of 3 are 3, 1.
The factors of 4 are 4, 2, 1.
The factors of 5 are 5, 1.
The factors of 6 are 6, 3, 2, 1.
The factors of 7 are 7, 1.
The factors of 8 are 8, 4, 2, 1.
The factors of 9 are 9, 3, 1.
The factors of 10 are 10, 5, 2, 1.
The factors of 11 are 11, 1.
The factors of 12 are 12, 6, 4, 3, 2, 1.

The factors of 13 are 13, 1.
The factors of 14 are 14, 7, 2, 1.
The factors of 15 are 15, 5, 3, 1.
The factors of 16 are 16, 8, 4, 2, 1.
The factors of 17 are 17, 1.
The factors of 18 are 18, 9, 6, 3, 2, 1.
The factors of 19 are 19, 1.
The factors of 20 are 20, 10, 5, 4, 2, 1.
The factors of 21 are 21, 7, 3, 1.
The factors of 22 are 22, 11, 2, 1.
The factors of 23 are 23, 1.
The factors of 24 are 24, 12, 8, 6, 4, 3, 2, 1.
The factors of 25 are 25, 5, 1.
The factors of 26 are 26, 13, 2, 1.
The factors of 27 are 27, 9, 3, 1.
The factors of 28 are 28, 14, 7, 4, 2, 1.
The factors of 29 are 29, 1.
The factors of 30 are 30, 15, 10, 6, 5, 3, 2, 1.
The factors of 31 are 31, 1.
The factors of 32 are 32, 16, 8, 4, 2, 1.

**b.** 1 has only one factor, itself, so it is not regarded as a prime number.

**c.** The numbers shaded on the board below are primes between 1 and 100. (The only even prime number is 2. Apart from the number 5, all other prime numbers end in 1, 3, 7, or 9.)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **2** | **3** | 4 | **5** | 6 | **7** | 8 | 9 | 10 |
| **11** | 12 | **13** | 14 | 15 | 16 | **17** | 18 | **19** | 20 |
| 21 | 22 | **23** | 24 | 25 | 26 | 27 | 28 | **29** | 30 |
| **31** | 32 | 33 | 34 | 35 | 36 | **37** | 38 | 39 | 40 |
| **41** | 42 | **43** | 44 | 45 | 46 | **47** | 48 | 49 | 50 |
| 51 | 52 | **53** | 54 | 55 | 56 | 57 | 58 | **59** | 60 |
| **61** | 62 | 63 | 64 | 65 | 66 | **67** | 68 | 69 | 70 |
| **71** | 72 | **73** | 74 | 75 | 76 | 77 | 78 | **79** | 80 |
| 81 | 82 | **83** | 84 | 85 | 86 | 87 | 88 | **89** | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | **97** | 98 | 99 | 100 |

### Activity Two

The sieve should give you all the primes found on the hundreds board.

### Investigation One

You have already found all the primes to 100 and discovered that apart from 2 and 5, all the primes end in 1, 3, 7, or 9. The next primes after 7 are 11 and 13. The most efficient way to find the primes between 101 and 200 is to sieve out any multiples of 2, 3, 5, 7, 11, or 13. New instructions might be:

6. Circle 11 because it is prime, and then cross out all the multiples of 11 except 11 itself.
7. Circle 13 because it is prime, and then cross out all the multiples of 13 except 13 itself.
8. Now circle all the numbers that have not been circled or crossed out.

Following all the instructions for numbers between 101 and 200 will give you the extra primes in this range. Finding the multiples of 11 and 13 also covers off the only multiple of 17 in this range that is a prime (17 x 11 = 187). Multiples up to 200 of primes larger than 17 are already covered off by finding the multiples of 2, 3, 5, and 7.

The shaded numbers below are the primes between 101 and 200 inclusive:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **101** | 102 | **103** | 104 | 105 | 106 | **107** | 108 | **109** | 110 |
| 111 | 112 | **113** | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 121 | 122 | 123 | 124 | 125 | 126 | **127** | 128 | 129 | 130 |
| **131** | 132 | 133 | 134 | 135 | 136 | **137** | 138 | **139** | 140 |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | **149** | 150 |
| **151** | 152 | 153 | 154 | 155 | 156 | **157** | 158 | 159 | 160 |
| 161 | 162 | **163** | 164 | 165 | 166 | **167** | 168 | 169 | 170 |
| 171 | 172 | **173** | 174 | 175 | 176 | 177 | 178 | **179** | 180 |
| **181** | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 |
| **191** | 192 | **193** | 194 | 195 | 196 | **197** | 198 | **199** | 200 |

### Investigation Two

**1.** **a.** No. Many palindromic numbers are even numbers or end in 5 and are therefore multiples of 2 or 5. Some palindromic primes are: 131, 353, 373, 383, 727, 757, 787, and 919.

**b.** Some 5-digit palindromic primes are: 10301, 11411, 12421, 14741, 95959.

**2.** **a.** The 8 emirps on the hundreds board are: 13, 31, 17, 71, 37, 73, 79, 97. There are 28 three-digit emirps, including 107, 113, 149, and 157. (Try an Internet search under "emirps".)

**b.** Some 4-digit emirps are: 1009, 1021, 1031, 1033, 1061, 1069.

## Pages 16–18: Prime Time

### Activity One

1. The other possible 36-pot rectangular trays are: 1 by 36, 2 by 18, 3 by 12, and 6 by 6. However, 1 by 36 and 2 by 18 are unlikely in practice. (Note that the square 6 by 6 tray is also rectangular because it fits the definition of a rectangle: it has 4 right angles and opposite sides are equal.)

2. Other equal packets are: 1 pkt of 56, 2 pkts of 28, 4 pkts of 14, 8 pkts of 7, 14 pkts of 4, 28 pkts of 2, and 56 pkts of 1. However, some packets, such as 1 pkt of 56 and 56 pkts of 1, are unlikely in practice. (The factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56.)

3. The other possible boxes are: 1 x 1 x 84, 1 x 2 x 42, 1 x 4 x 21, 1 x 6 x 14, 1 x 3 x 28, 1 x 7 x 12, 2 x 14 x 3, 2 x 2 x 21, 3 x 4 x 7.

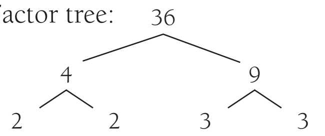
### Activity Two

1. Paora's method is different from the others because he begins by using 6 and 10 as his first factors. Holly and Amanda both start with 2 and 30 as factors. Holly then splits 30 as 2 x 15. Amanda splits 30 as 3 x 10.

2. Yes, because prime factors cannot be separated into two other factors. Other sets of factors for 60, for example, 2 x 2 x 15, include one or more factors that are not prime factors.

3. Strategies will vary. For example, you could use factor trees or division.

Factor tree:




Division:

2)36
2)18
3) 9
3

Prime factors are:

a. 36 = 2 x 2 x 3 x 3

b. 54 = 2 x 3 x 3 x 3

c. 72 = 2 x 2 x 2 x 3 x 3

d. 84 = 2 x 2 x 3 x 7

e. 140 = 2 x 2 x 5 x 7

f. 210 = 2 x 3 x 5 x 7.

### Activity Three

1. a. Jane saw that there was an extra factor of 2 in the denominator, so the whole fraction is equivalent to $\frac{1}{2}$.

b. i. $\frac{54}{72} = \frac{2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 3 \times 3} = \frac{3}{2 \times 2} = \frac{3}{4} = 75\%$

ii. $\frac{60}{72} = \frac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 2 \times 3 \times 3} = \frac{5}{6} = 83.\dot{3}\%$

iii. $\frac{63}{72} = \frac{3 \times 3 \times 7}{2 \times 2 \times 2 \times 3 \times 3} = \frac{7}{8} = 87.5\%$

2. a. $\frac{210}{30} = \frac{2 \times 3 \times 5 \times 7}{2 \times 3 \times 5} = 7$ hrs

b. $\frac{210}{42} = \frac{2 \times 3 \times 5 \times 7}{2 \times 3 \times 7} = 5$ hrs

c. $\frac{210}{84} = \frac{2 \times 3 \times 5 \times 7}{2 \times 2 \times 3 \times 7} = \frac{5}{2} = 2\frac{1}{2}$ hrs

d. $\frac{210}{60} = \frac{2 \times 3 \times 5 \times 7}{2 \times 2 \times 3 \times 5} = \frac{7}{2} = 3\frac{1}{2}$ hrs

e. $\frac{210}{140} = \frac{2 \times 3 \times 5 \times 7}{2 \times 2 \times 5 \times 7} = \frac{3}{2} = 1\frac{1}{2}$ hrs

3. Yes. They can have 6 teams made up of 6 guides and 9 scouts. Using prime factors:
36 = 2 x (2 x 3) x 3 = 6 x 6
54 = 2 x 3 x (3 x 3) = 6 x 9

### Activity Four

1. a. 1 ÷ 8 = 0.125

b. 1 ÷ 20 = 0.05

c. 1 ÷ 125 = 0.008

d. 1 ÷ 64 = 0.015625

e. 1 ÷ 100 = 0.01

So, yes, all these decimals terminate.

f. 1 ÷ 36 = $0.02\dot{7}$

g. 1 ÷ 60 = $0.01\dot{6}$

h. 1 ÷ 54 = $0.0\overline{185}$

i. 1 ÷ 140 = $0.00\overline{714285}$

j. 1 ÷ 210 = $0.00\overline{476190}$

None of the divisions in this group terminate.

2. a. In each case where the decimals terminate, the divisor has no prime factors other than 2 and/or 5.

b. If the divisor has any prime number other than 2 or 5 as a factor, the decimal will not terminate.

### Challenge

If a number other than 1 is involved in the division, the prime factors of the number being divided must be taken out of the divisor's prime factors before the check for factors other than 2 and 5 is done. For example, with 3 ÷ 120, the prime factors of 120 are 2 x 2 x 2 x 3 x 5. Taking out 3 as a factor leaves only 2 x 2 x 2 x 5, so the decimal will terminate. For 7 ÷ 22, the prime factors of 22 are 2 and 11. There is no 7 (the prime factor in the number being divided) to take out, but because 11 is a prime number other than 2 or 5, the decimal will not terminate.

## Page 19: Fred's Rent-a-Fence

### Activity

1. a. 5 different rectangular shapes are: 1 m by 80 m, 2 m by 40 m, 4 m by 20 m, 5 m by 16 m, and 8 m by 10 m. Only the last two shapes would suit the dodgems (the other three would be too narrow).

   b. The 8 m by 10 m rectangle would use 36 panels and cost $108, which is cheaper than the other options. It is one of the shapes that would suit the dodgems.

2. a. There are 12 different-sized rectangles that could be made.

| Width | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Length | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 |

   b. The largest option is 12 m wide and 12 m long, which gives an area of 144 $m^2$. The shape of this area is a square.

3. 50 m. The length must be 15 m because 10 x 15 = 150. 2 x (10 + 15) = 2 x 25 = 50 gives the perimeter.

### Investigation

Answers may vary. There are three whole-number solutions: 4 x 4, 3 x 6, and 6 x 3 (which is the same as 3 x 6). There is an infinite number of solutions if rectangles with only one whole-number side or no whole-number sides are included.

## Pages 20–21: Dotty People

### Activity

1. 10 000 (ten thousand).
   (10 x 10 = 100 blocks; 1 block has 100 dots, so 100 x 100 = 10 000 dots.)

2. a. Mangaiti 12 000 (twelve thousand)

   b. Tapanui 2 200 (two thousand, two hundred)

   c. Waikakariki 2 800 (two thousand, eight hundred)

   d. Ngāpuke 3 900 (three thousand, nine hundred)

3. a. (Multiply each population by 10.)
   Mangaiti 120 000
   Tapanui 22 000
   Waikakariki 28 000
   Ngāpuke 39 000

   b. (Multiply each population by 100.)
   Mangaiti 1 200 000
   Tapanui 220 000
   Waikakariki 280 000
   Ngāpuke 390 000

   c. (Halve the answers to **b**.)
   Mangaiti 600 000
   Tapanui 110 000
   Waikakariki 140 000
   Ngāpuke 195 000

4. a.–e. Dot arrays will vary, depending on the population figures used and how many people each dot represents. Approximate populations (2006 figures in brackets) are:
   Auckland: about 1.3 million (1 337 000)
   New Zealand: about 4 million (4 130 000)
   Sāmoa: about 170 thousand (177 000)
   Australia: about 20 million (20 264 000)
   World: about 6.5 billion (6 525 170 000).

## Page 22–24: Rolling Up!

### Activity

1. a. Yes. ($\frac{4}{8} = \frac{1}{2}$, so $\frac{3}{8}$ is less than $\frac{1}{2}$)

   b. No. ($\frac{1\frac{1}{2}}{3} = \frac{1}{2}$, so $\frac{2}{3}$ is more than $\frac{1}{2}$)

   c. No. ($\frac{5}{10} = \frac{1}{2}$, so $\frac{6}{10}$ is more than $\frac{1}{2}$)

   d. No. ($\frac{3\frac{1}{2}}{7} = \frac{1}{2}$, so $\frac{4}{7}$ is more than $\frac{1}{2}$)

   e. Yes. ($\frac{5\frac{1}{2}}{11} = \frac{1}{2}$, so $\frac{5}{11}$ is less than $\frac{1}{2}$)

   f. No. ($\frac{6\frac{1}{2}}{13} = \frac{1}{2}$, so $\frac{7}{13}$ is more than $\frac{1}{2}$)

   g. Yes. ($\frac{14\frac{1}{2}}{29} = \frac{1}{2}$, so $\frac{14}{29}$ is less than $\frac{1}{2}$)

   h. No. ($\frac{49}{98} = \frac{1}{2}$)

2. a. Rolf: The double the numerator method works and is easy to use.

   b. Moana: Going through the list of equivalent fractions is slow and cumbersome. With odd denominators, a different check is needed, such as method **a**.

c. Jake: The halve the denominator method works and is easy to use.

3. a. i. Yes. ($\frac{9}{12} = \frac{3}{4}$, so $\frac{10}{12}$ is more than $\frac{3}{4}$.)

ii. No. ($\frac{75}{100} = \frac{3}{4}$, so $\frac{74}{100}$ is less than $\frac{3}{4}$.)

iii. Yes. ($\frac{24}{32} = \frac{3}{4}$)

iv. No. ($\frac{11}{45}$ is less than $\frac{1}{4}$, so $\frac{33}{45}$ is less than $\frac{3}{4}$.)

b. Answers will vary, but these methods will work:

i. Divide the denominator by 4, then multiply the result by 3. If the answer is less than the numerator, then the fraction is greater than $\frac{3}{4}$.

ii. Divide the numerator by 3 and multiply the result by 4. If the answer is greater than the denominator, then the fraction is greater than $\frac{3}{4}$.

4. Piripi:
$\frac{7}{12}$: 12 ÷ 3 = 4, 4 x 2 = 8. $\frac{8}{12} = \frac{2}{3}$, so $\frac{7}{12}$ is less than $\frac{2}{3}$.
$\frac{10}{14}$: 14 ÷ 3 = $4\frac{2}{3}$, $4\frac{2}{3}$ x 2 = $9\frac{1}{3}$. $\frac{9\frac{1}{3}}{14} = \frac{2}{3}$, so $\frac{10}{14}$ is greater than $\frac{2}{3}$.

Marissa:
$\frac{7}{12}$: 7 ÷ 2 = $3\frac{1}{2}$, $3\frac{1}{2}$ x 3 = $10\frac{1}{2}$. $\frac{7}{10\frac{1}{2}} = \frac{2}{3}$, so $\frac{7}{12}$ is less than $\frac{2}{3}$.
$\frac{10}{14}$: 10 ÷ 2 = 5, 5 x 3 = 15. $\frac{10}{15} = \frac{2}{3}$, so $\frac{10}{14}$ is greater than $\frac{2}{3}$.

5. a. Greater than. $\frac{22}{33} = \frac{2}{3}$, so $\frac{23}{33}$ is greater than $\frac{2}{3}$.

b. Equal to. (Both 30 and 45 divide by 15. That is, Piripi's and Marissa's methods both give $\frac{30}{45} = \frac{2}{3}$.)

6. a. i.–ii. The statements are true for all four numbers. (For example,
12 x 3 ÷ 4 = 36 ÷ 4 = 9, and
12 ÷ 4 x 3 = 3 x 3 = 9;
12 x 2 ÷ 3 = 24 ÷ 3 = 8, and
12 ÷ 3 x 2 = 4 x 2 = 8.)

b. Yes, the statements work for any numbers. Changing the order of the multiplication and division does not change the answer.

### Challenge

Possible methods include:

- using equivalent fractions and dividing
- converting all the fractions to decimals.

a. $\frac{1}{2}$ length rolls: 11. $\frac{23}{4} = \frac{11\frac{1}{2}}{2}$, so 11 half-length rolls can be made.
$\frac{2}{3}$ length rolls: 8. $\frac{2}{3} = \frac{8}{12}$ and $\frac{23}{4} = \frac{69}{12}$;
69 ÷ 8 = 8 r5, so 8 two-third-length rolls are possible.
$\frac{3}{4}$ length rolls: 7. $\frac{23}{4} \div \frac{3}{4}$ = 23 ÷ 3 = 7 r2, so 7 three-quarter-length rolls can be made.

b. $\frac{1}{2}$ length rolls: 8. $\frac{13}{3} = 4\frac{1}{3}$. $4 \div \frac{1}{2} = 8$, so 8 half-length rolls can be made.
$\frac{2}{3}$ length rolls: 6. $\frac{13}{3} \div \frac{2}{3} = 6\frac{1}{2}$, so 6 two-third-length rolls can be made.
$\frac{3}{4}$ length rolls: 5. $\frac{13}{3} = 4\frac{1}{3}$. There are 16 quarters in 4 ($\frac{16}{4}$ = 4), so 5 lots of $\frac{3}{4}$ can be made from 4 lengths (5 x 3 = 15). The extra $\frac{1}{4} + \frac{1}{3}$ is less than $\frac{3}{4}$.

c. $\frac{1}{2}$ length rolls: 15. $\frac{38}{5} = 7\frac{3}{5}$. From $7\frac{1}{2}$, 15 half-length rolls can be made.
$\frac{2}{3}$ length rolls: 11. $\frac{2}{3} = \frac{10}{15}$ and $\frac{38}{5} = \frac{114}{15}$.
114 ÷ 10 = 11.4, so 11 two-third-length rolls can be made.
$\frac{3}{4}$ length rolls: 10. $7\frac{1}{2} = \frac{15}{2} = \frac{30}{4}$, so 10 lengths of $\frac{3}{4}$ can be made.

d. $\frac{1}{2}$ length rolls: 16. $\frac{59}{7}$ is $8\frac{3}{7}$, which is just less than $8\frac{1}{2}$. From $8\frac{1}{2}$, 17 half-length rolls can be made, so from $8\frac{3}{7}$, 16 half-length rolls can be made.
$\frac{2}{3}$ length rolls: 12. Based on $8\frac{3}{7}$ being just less than $8\frac{1}{2}$: $\frac{2}{3} = \frac{4}{6}$ and $8\frac{1}{2} = \frac{17}{2} = \frac{51}{6}$. 51 ÷ 4 = $12\frac{3}{4}$, so 12 two-third-length rolls can be made.
$\frac{3}{4}$ length rolls: 11. Based on $8\frac{3}{7}$ being just less than $8\frac{1}{2}$: $8\frac{1}{2} = \frac{34}{4}$, 34 ÷ 3 = $11\frac{1}{3}$, so 11 three-quarter-length rolls can be made.

e. $\frac{1}{2}$ length rolls: 24. $\frac{99}{8} = 12\frac{3}{8}$. 12 x 2 = 24
$\frac{2}{3}$ length rolls: 18. $\frac{99}{8} = 12\frac{3}{8}$. ($\frac{3}{8}$ is less than $\frac{2}{3}$.)
12 = $\frac{36}{3}$; 36 ÷ 2 = 18.
$\frac{3}{4}$ length rolls: 16. $\frac{99}{8} = 12\frac{3}{8}$. ($\frac{3}{8}$ is less than $\frac{3}{4}$.)
12 = $\frac{48}{4}$. 48 ÷ 3 = 16.

f. $\frac{1}{2}$ length rolls: 32. $\frac{146}{9} = 16\frac{2}{9}$, so 32 half-length rolls can be made.
$\frac{2}{3}$ length rolls: 24. $\frac{2}{3} = \frac{6}{9}$; $\frac{146}{9} \div \frac{6}{9} = 24\frac{1}{3}$ (or 146 ÷ 6 = $24\frac{1}{3}$), so 24 two-third-length rolls can be made.
$\frac{3}{4}$ length rolls: 21. $\frac{146}{9} = 16\frac{2}{9}$. From 15, 20 three-quarter-length rolls can be made. (15 = $\frac{60}{4}$. $\frac{60}{4} \div \frac{3}{4}$ = 20) From the remaining $1\frac{2}{9}$, one extra $\frac{3}{4}$ can be made, making a total of 21 strips.

Figure It Out

## Multiplicative Thinking

# Teachers' Notes

## Overview: Levels 3–4

# Introduction to Multiplicative Thinking

Multiplicative thinking is the term used to describe thinking that employs the mathematical power of multiplication. Students need to use the properties of multiplication in order to understand many areas of mathematics, such as area and volume, the metric measurement system, fractions, and algebra.

The Number Framework outlines the way that children seem to build their multiplicative understandings on their additive ones, just as they previously built additive understandings out of their knowledge of counting. Understanding multiplication as repeated addition seems to be an important step for many students and is often a starting point for instruction.

Many problems are multiplicative, and while the most commonly used strategy is repeated addition or equal groups, we also need to include other models of multiplication to extend and deepen students' understanding. The Figure It Out *Multiplicative Thinking* books include activities that explore multiplication as a comparison, a rate, an array, and a Cartesian product. From each of these problem types, students will begin to extract the key ideas of multiplication as commutative, distributive, and associative. Understanding how multiplication works helps students to use the strategies outlined in the Numeracy Development Project booklets and in these Figure It Out books.

Division is the inverse of multiplication in the same way as subtraction is the inverse of addition. Traditionally regarded as the most difficult of the operations to understand and perform, division, research suggests, is best learned alongside multiplication. Situations involving division can sometimes be solved using multiplication, particularly when the numbers are in the basic facts range. Recognising the relationship between multiplication and division is a very important task for students.

To work effectively with multiplicative strategies, students need to know their basic facts. In the initial stages, they will use skip-counting and materials to develop a conceptual understanding of what multiplication statements mean. This emerging concept feeds into, and is fed from, knowledge of basic facts. Recall of basic multiplication facts does not equate with multiplicative thinking, but multiplicative thinking will be facilitated by a sound knowledge of these facts.

A change in students' thinking is needed if they are to fully understand multiplication. In multiplication, they encounter for the first time numbers that are telling them to perform an operation rather than numbers that represent an amount. For example, in 4 x 3 (which can be read as four sets of 3 or as three sets of 4), one number tells how many times the other should be repeated rather than representing an amount (a set of 4 plus a set of 3). However, solving multiplication problems additively like this can be unwieldy; students who continue to use this method will have problems with more sophisticated concepts.

The activities in the Figure It Out *Multiplicative Thinking* books use a range of problem types, consider strategies based on the properties of multiplication, ask students to explore these properties, and include division as the inverse of multiplication. In doing so, the books aim to support teachers in developing a robust and deep understanding of multiplication in their learners.

# Page 1: RJ's Fabulous Juice Bar

**Achievement Objectives**

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find fractions equivalent to one given (Number, level 4)
- find a given fraction or percentage of a quantity (Number, level 4)

AC
EA
AA
AM
AP

**Number Framework Links**

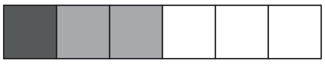
Students learn to duplicate ratios when they are moving from using advanced additive strategies (stage 6) towards using advanced multiplicative strategies (stage 7). To find difficult equivalent ratios, students need to be operating at least at the advanced multiplicative stage.

## Activity

In this activity, students use equivalent ratios in the context of mixing juice cocktails.

In this context, a ratio is a short, symbolic way of describing the relative amounts of the different juices used to make a cocktail. A cocktail made with 1 part of A, 2 parts of B, and 3 parts of C is described by the ratio 1:2:3. All parts of a ratio are measured using the same unit (in this case, millilitres [mL]), but the ratio works regardless of the unit.

It is important to distinguish between ratios, which involve part-to-part comparisons, and proportions, which involve part-to-whole comparisons. You could discuss the difference between the two with reference to this 1:2:3 ratio:



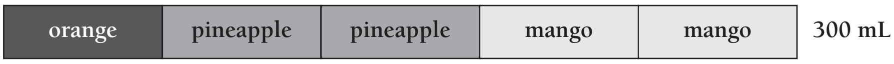
Only 1 of the 6 parts is black, so the part-to-whole comparison for black is $\frac{1}{6}$, while for grey it is $\frac{2}{6}$ or $\frac{1}{3}$, and for white it is $\frac{3}{6}$ or $\frac{1}{2}$.

In an equivalent ratio, the amounts of each component part of the ratio are increased or decreased without changing these part-to-part and part-to-whole comparisons. Have the students imagine the 1:2:3 ratio doubled:

The ratio is now 2:4:6, and the part-to-whole comparisons are $\frac{2}{12}$ or $\frac{1}{6}$ black, $\frac{4}{12}$ or $\frac{1}{3}$ grey, and $\frac{6}{12}$ or $\frac{1}{2}$ white. The students should note that the comparisons remain preserved when a ratio is duplicated or split into equal amounts. This fact, known as conservation of ratio, is not always an easy one for students to grasp.

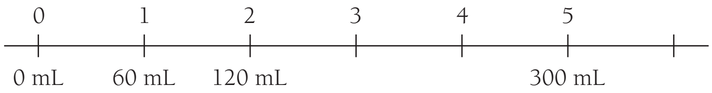
Strip diagrams, ratio tables, and double number lines have been shown to be excellent tools for assisting students to think through problems involving ratios and proportions. This activity provides a good context for developing students' understanding and skill with these tools.

Here is question **1a** represented as a strip diagram:




The diagram shows clearly that the amount of orange is $\frac{1}{5}$ of 300 (60 mL) and that the amount of pineapple and mango is $\frac{2}{5}$ of 300 (120 mL) each.
Here is the question represented with interlocking cubes of three patterns:

The students could illustrate the problem using a double number line:




Alternatively, the information could be set out in a ratio table like this:

| **5 parts** | **1 part** | **2 parts** |
|---|---|---|
| **300 mL** | **60 mL** | **120 mL** |

Note that the strip diagram is the least abstract of these representations.

Help the students to model question **1a** as above but encourage them to draw their own diagrams to represent the other problems. Be aware that some may be uncertain of measurement conversions, for example, 1 L = 1 000 mL.

Here is one way of using strip diagrams to visualise the comparison in question **1b**:

**Medium Hawaiian**

| orange | pineapple | pineapple | mango | mango | **600 mL** |
|---|---|---|---|---|---|

**Medium Pineapple Punch**

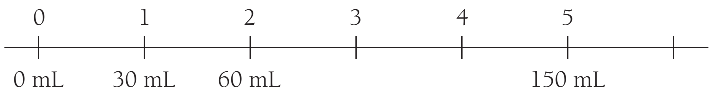
| pineapple | pineapple | apple | **600 mL** |
|---|---|---|---|

The 2 parts of pineapple in a medium Hawaiian amount to 2 x 120 = 240 mL. 1 part of pineapple in the medium Pineapple Punch is 200 mL, so 2 parts is 400 mL. The 400 mL in the second drink is clearly more than the 240 mL in the first. Alternatively, $\frac{2}{5}$ is less than $\frac{2}{3}$, so a Pineapple Punch needs more pineapple juice than a Hawaiian.

Here is a strip diagram for question **1c**:

| coconut milk | mango | mango | orange | **1000 mL (1 L)** |
|---|---|---|---|---|
| **250 mL** | **250 mL** | **250 mL** | **250 mL** | |

Any number less than 300 mL (the size of a small drink) is a possibility for the tiny drink called for in question **3a**. For convenience, the number should be a multiple of 5 so that it can be easily divided into 5 parts as required by the ratio. Better still, if the number is a multiple of 50, each part will be a multiple of 10 mL, which will give sensible and convenient amounts for each of the three component juices. If 150 mL is chosen for the tiny size, the information can be represented in this way:




1:2:2 is equivalent to 30:60:60, so a tiny Hawaiian will have 30 mL of orange and 60 mL of both pineapple and mango.

For the giant drink in question **3b**, any size greater than 1 L and less (perhaps) than 1.5 L is possible. If the size is a multiple of 40, each part will be a multiple of 10 mL.

**Links**

**Figure It Out**

- *Number: Book One,* Levels 3–4
  Stretch and Grow, pages 4–5
  Bean Brains, page 9
- *Number Sense and Algebraic Thinking: Book One,* Level 3
  Run like the Wind, pages 12–13

- *Number Sense and Algebraic Thinking: Book Two,* Levels 3–4
  Lunchtime Mardi Gras, pages 18–20
- *Number Sense: Book Two,* Years 7–8, Level 4
  Balancing Act, pages 22–23
- *Number: Book Five,* Years 7–8, Level 4
  Bargain Packs, page 15
- *Number: Book Six,* Years 7–8, Level 4+
  Hypertufa Tiles, page 17
- *Proportional Reasoning*, Level 3+
  Chocolate Choices, page 4
  Pop Star Pics, page 20
- *Proportional Reasoning: Book One,* Levels 3–4
  Smart Sizes, page 21
  The Right Gear, page 20

## Pages 2–3: Divided We Stand

**Achievement Objectives**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- write and solve problems involving decimal multiplication and division (Number, level 4)

AC
EA
AA
AM
AP

**Number Framework Links**

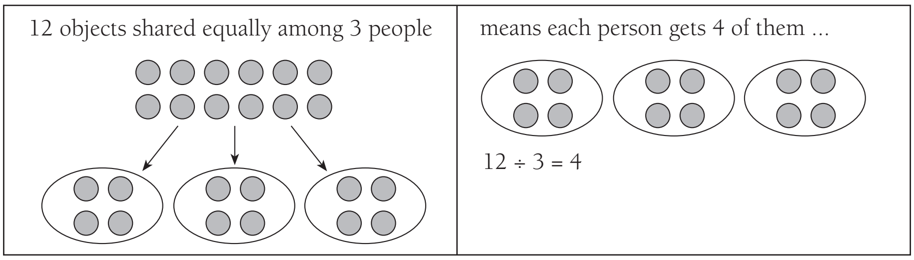
Students already using advanced additive strategies (stage 6) learn about splitting dividends and divisors multiplicatively when they are moving towards using advanced multiplicative strategies (stage 7).

### Activity

This activity focuses on the number properties that are useful in solving equal-sharing and equal-sets division problems.

The introductory example starts with a total set of 72 objects and shows two possible ways of arranging the objects in smaller sets. Students will know how to express relationships of this kind as multiplications, in the form 9 x 8 = 72. In this activity, they learn that such relationships may equally well be expressed as divisions, in the form 72 ÷ 8 = 9.

You may find it useful to get your students to work initially with a smaller total set (such as 12) and to consider ways of sharing the set equally. For example:




More generally, we can say that if there are *a* objects shared equally among *b* people, each person gets *c* of them ($a \div b = c$).

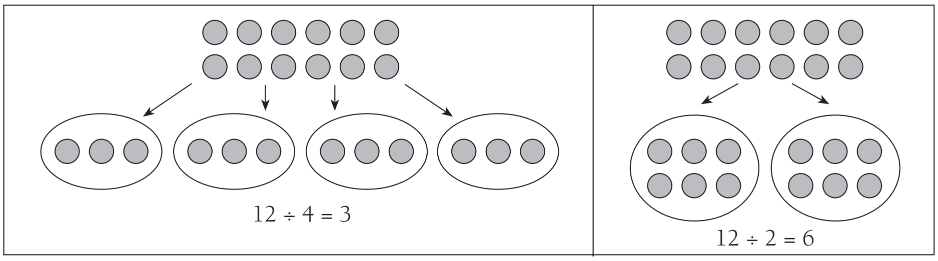
The picture also shows that 3 sets of 4 can be made from 12 objects. The answer to the question "How many sets of 4 can be made from 12?" can be written as 12 ÷ 4 = 3, meaning, in terms of the picture above, that 3 people would each get a set of 4. (Alternatively, the answer to "How many sets of 3 can be made from 12?" can be written as 12 ÷ 3 = 4.) It is very important that students learn that the division sign can be interpreted as either "split equally between" or "split equally into sets of". Using algebraic notation, we can say that if $a \div b = c$, then $a \div c = b$.

In summary, and using a set of 40 this time, if 40 ÷ 8 = 5, then 40 ÷ 5 = 8, 8 x 5 = 40, and 5 x 8 = 40. Expressed algebraically, if $a \div b = c$, then these three other statements are also true: $a \div c = b$, $b \times c = a$, and $c \times b = a$.

In parts **a** and **b** of question **1**, students explore the important relationship discussed above, and in part **c** they are reminded that an inverse relationship exists between the number of sets and the number of objects in each set. They will find this relationship useful when doing the later questions.

In question **2**, students explore some strategies for making division calculations. You could introduce the ideas using examples such as the following:

Consider the connection between 12 ÷ 4 = 3 and 12 ÷ 2 = 6 as cases of equal sharing:




When 12 objects are shared into 2 equal sets, those sets contain twice as many objects as would be the case when 12 objects are shared into 4 equal sets.

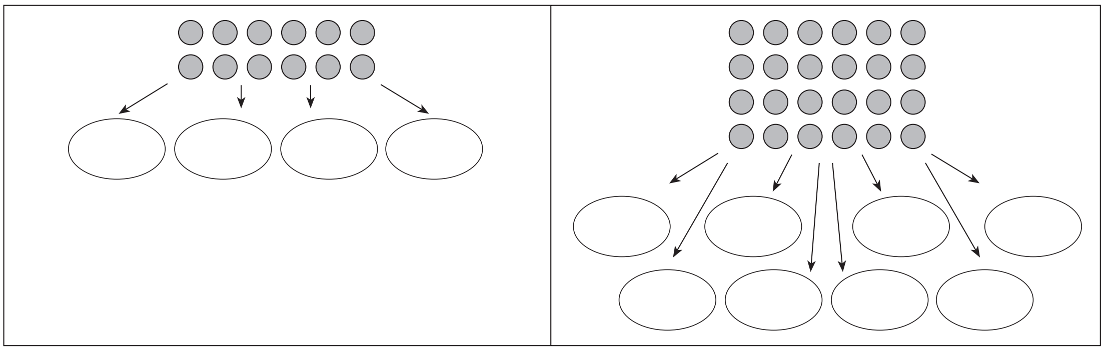
If these sharing cases are written as multiplications, 4 x 3 = 12 and 2 x 6 = 12, students should recognise the doubling and halving that is going on here.

Strategies that make use of inverse relationships (especially doubling and halving and thirding and trebling strategies) can be used to simplify many division problems. For example, if there are *a* objects split equally into *b* groups and the number of objects in each group is *c*:

- halving the number of groups doubles the number of objects in each group ($a \div \frac{b}{2} = 2c$)
- thirding the number of groups trebles the number of objects in each group ($a \div \frac{b}{3} = 3c$).

It is also possible to multiply or divide both the total number of objects *a* and the divisor *b* by the same amount without affecting the result. This fact can be used as a strategy for making division problems easier: the problems 48 ÷ 16 = ☐, 24 ÷ 8 = ☐, and 12 ÷ 4 = ☐ all have the same answer, 3.

To help your students see the reason for this, have them explore the following pair of diagrams. Ask *How are the diagrams related, and how many objects will end up in each group in each case?*



If 12 objects are shared equally into 4 groups, there will be 3 objects in each. If twice as many objects are shared into twice as many groups, there will still be 3 objects in each. Expressed algebraically, if $a \div b = c$, then $2a \div 2b = c$, $3a \div 3b = c$, and so on.

However, the multiplier for both numbers (objects and groups) doesn't have to be a whole number. It can be a fraction, with the result that both numbers get smaller and the answer is easier to calculate. For example, 72 ÷ 24 = □ has the same answer as 12 ÷ 4 = □. Both 72 and 24 have been multiplied by one-sixth, which is the same as dividing both by 6. Expressed algebraically, if $a \div b = c$, then $\frac{a}{2} \div \frac{b}{2} = c$, $\frac{a}{3} \div \frac{b}{3} = c$, and so on.

Question **3** is ideally suited to discussion in small groups. The students think about the thought processes being used and then replicate those processes using their own choice of numbers.

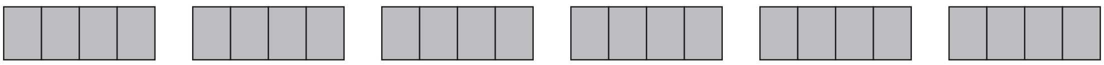
Question **4** deliberately uses larger numbers to ensure that students will have to simplify them using a strategy of the kind described above.

Question **5** provides a check that the students really understand what they are doing.

***Imaging***

It is important that students understand the structure behind the number properties rather than just recognise and follow patterns. Have them represent each division problem, using materials so that they make the connections between words, symbols, and quantities. Interlocking cubes are ideal for this purpose.

For example, here is a model for 24 ÷ 6 = 4 as "equal sharing":



Starting with this 24-cube model, have your students pose new sharing questions (predictive questions) and then anticipate how they would need to alter the blocks to model the new outcome. For example, "How would I change this into 24 ÷ 3 = □?" "What would 24 ÷ 12 = □ look like?" Ensure that your students record the results with a diagram and an equation.

Asking themselves predictive questions encourages students to image the materials. Imaging is an important bridge between the concrete (materials) and the abstract (number properties).

***Taking the ideas further***

While simple examples like this are useful in the Using Materials and Using Imaging stages of the Numeracy Development Project (NDP) teaching model, they are unlikely to sell the power of the number properties to students. It is vital that the difficulty level is raised to the point where the power of the number properties becomes obvious and is essential in solving calculations.

For example, by posing the problem *If there are 144 lollies to be shared among 16 people, how many lollies does each person get?*, you will force the students to choose a strategy that will make the problem manageable. In this case, halving both numbers reduces the problem's complexity to the point where it can be solved by the use of basic facts (144 ÷ 16 = 72 ÷ 8 = 9).

By carefully selecting the numbers involved in problems, you can ensure that some strategies will be more efficient than others. Efficiency relates to the number and complexity of the steps involved in the calculation. For example, the problem 252 ÷ 6 = □ might be solved in either of these two ways:
either 240 ÷ 6 = 40; 252 – 240 = 12, 12 ÷ 6 = 2; 40 + 2 = 42;
or 252 ÷ 6 = 126 ÷ 3 = 42.

Although both strategies use place value understanding, the second (halving both numbers) is clearly more efficient. It is important that students explore which strategy or strategies is/are most efficient for any given problem.

Another useful extension question is 279 ÷ 9. Students may initially look at using one of the strategies in the activity but will hopefully recognise that in this case, place value partitioning is best: 270 ÷ 3 = 30; 9 ÷ 9 = 1; 30 + 1 = 31, so 279 ÷ 9 = 31. By inspection, they know that 279 is divisible by 9 because the digital root is 9 (2 + 7 + 9 = 18; 1 + 8 = 9).

**Links**

**Numeracy Project materials** (see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 6: Teaching Multiplication and Division*
  Little Bites, page 42
  Royal Cooking Lesson, page 30
  Proportional Packets, page 28

**Figure It Out**

- *Number Sense: Book One*, Years 7–8, Link
  Division Dilemmas, page 24

## Pages 4–5: Cutting It

**Achievement Objectives**

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

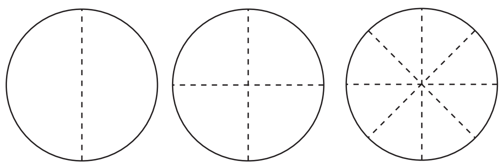
**Number Framework Links**

Students learn to find equivalent fractions multiplicatively while in transition from advanced additive (stage 6) to advanced multiplicative (stage 7).

### Activity

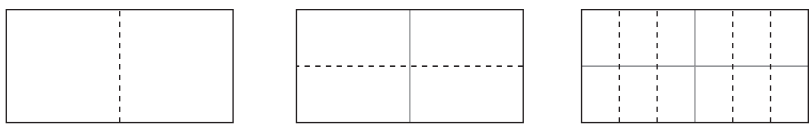
This activity is about fractions and division into equal parts. These concepts are strongly related. To partition an area into fractions, the parts must be equal.

Spatially, some shapes are harder to partition than others. As long as a shape is regular and has reflective symmetry, it can usually be partitioned very easily into halves, quarters, eighths, and so on by folding:



Have your students use repeated halving to establish that halving halves gives quarters (fourths) and halving quarters gives eighths. This has a division equivalent: a number can be divided by 4 by halving and then halving again. (For example, 72 ÷ 4 = □ can be solved as 72 ÷ 2 ÷ 2 = 18.) Similarly, a number can be divided by 8 by halving three times in succession, because 2 x 2 x 2 = 8. (For example, 144 ÷ 8 = □ can be solved as 144 ÷ 2 ÷ 2 ÷ 2 = 18.)

When doing question **1c**, students are likely to find that a good first step is to exhaust any halving opportunities before moving on to thirding, fifthing, or whatever. So, to cut a pie into twelfths, they might proceed like this:



Finding twelfths by halving, halving, and then thirding (based on the fact that 2 x 2 x 3 = 12), has a division equivalent. For example, 264 ÷ 12 = ☐ can be solved using these steps: 264 ÷ 2 = 132, 132 ÷ 2 = 66, and 66 ÷ 3 = 22, so 264 ÷ 12 = 22. Algebraically, if $b = q \times r \times s$, then $a \div b = a \div q \div r \div s$.

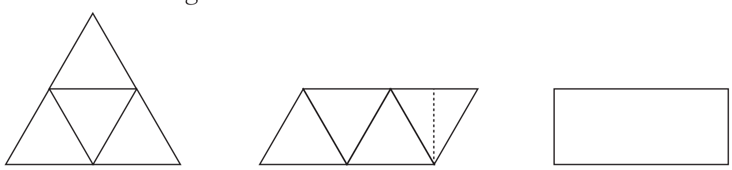
The learning point here is that many division problems can be solved more easily if they are done as a series of divisions by numbers that are factors of the divisor. For example, 972 ÷ 36 = 972 ÷ 2 ÷ 2 ÷ 3 ÷ 3 = 27 because 2 x 2 x 3 x 3 = 36.

It is also important that students understand that while partitioning can be thought of in terms of division, it can equally well be thought of in terms of multiplication by unit fractions (fractions with a numerator of 1). For example, the partitioning in question **1c ii** (pictured as a rectangle above) can be represented mathematically as $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$. Encourage your students to connect this with the fact that 2 x 2 x 3 = 12.

Some students struggle with the idea that multiplying by unit fractions (in fact, any fractions less than 1) gives them a result that is smaller than they started with. This is because they have generalised from whole numbers that multiplication "makes bigger". One way of correcting this misconception is to refer students back to the whole-number view of multiplication that sees 3 x 4 = ☐ as "3 sets of 4". It will then help if they can see $\frac{1}{2} \times \frac{1}{3} =$ ☐ as "one-half of one-third". Another useful approach is to use number patterns to help broaden the students' ability to generalise, for example, the pattern 4 x 3 = 12, 2 x 3 = 6, 1 x 3 = 3, $\frac{1}{2} \times 3 = 1\frac{1}{2}$, $\frac{1}{4} \times 3 = \frac{3}{4}$, and so on.

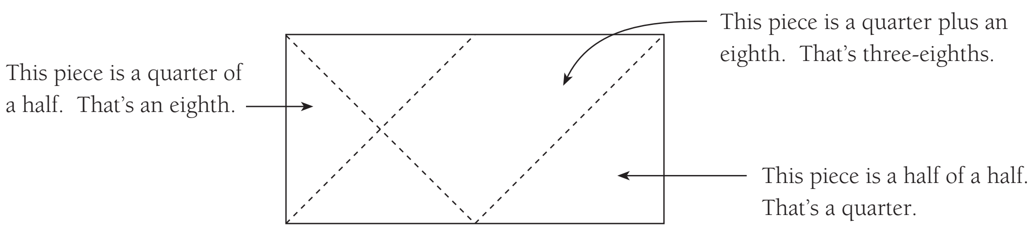
Questions **3–6** involve unequal partitioning. Question **3** is designed to challenge the misconception that, if there are 5 parts, each part must be $\frac{1}{5}$. It also leads on to an understanding that a whole (quiche) can be partitioned into 3 sixths and 2 quarters.

Students need to understand that when they partition a shape, area is conserved. That is, the sum of the areas of the parts equals the area of the whole. Discuss with your students how the triangle below is partitioned and the parts recombined to form a parallelogram that can then be partitioned to form a rectangle:



Question **4** requires the students to map parts back to the whole and to consider how many times a particular piece will fit in.

For example, for question **4d**:




***Taking the ideas further***

Students could also use this activity to explore the idea that equivalent fractions can be created by splitting an existing fraction. To understand equivalent fractions, they need to hold the "1 whole" in their heads while considering the effect of splitting a fractional part. For example, if a part is defined as $\frac{1}{3}$, then one-quarter of that part is $\frac{1}{12}$ of the original whole (1).

This means that $\frac{1}{3}$ is equivalent to $\frac{4}{12}$ of the same whole (that $\frac{1}{3} = \frac{4}{12}$). Students should see that the numerator and denominator of $\frac{1}{3}$ have both been multiplied by 4 because there are now 4 times as many parts making $\frac{1}{3}$ and 4 times as many parts making up the whole.

**Links**

**Numeracy Project materials** (see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Introduction, page 22
- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Equivalent Fractions, page 16

**Figure It Out**

- *Number: Book One,* Level 3
  Fun with Fractions, page 9
  More Fractions, page 10
  Racing to New Heights, page 14
- *Number: Book Three,* Level 3
  Fraction Frenzy, pages 22–23
- *Number: Book One,* Levels 3–4
  A Watery Mission, page 3
- *Number: Book Two,* Levels 3–4
  Sandwich Survey, page 11
- *Number Sense and Algebraic Thinking: Book One,* Level 3
  Fraction Tagging, pages 18–20
- *Number: Book Two,* Years 7–8, Link
  Boxed Biscuits, page 24
- *Proportional Reasoning: Book One,* Levels 3–4+
  Paper Partitions, page 6

## Pages 6–7: Fair Shares

**Achievement Objective**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)

AC | EA | AA | **AM** | AP

**Number Framework Links**

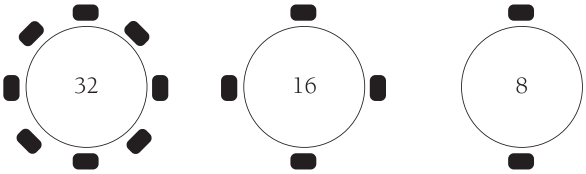
This activity can be used to encourage students at stage 7 to extend their range of strategies to include simplification by proportional adjustments.

### Activity

This activity deals with partitive division, that is, division where the number of parts is known but the amount of each share is not. Students learn that they can turn a complex division problem into a simpler one by simplifying both the dividend (the starting number) and the divisor (the number doing the dividing), using a common factor.

You could introduce the mathematics of question **1** with a simple example like the one below to illustrate the number properties. By doing this, you move the focus from the answer, which the students already know or can work out easily, to the number properties involved. By keeping the numbers manageable, you can use materials to demonstrate the transformations on the quantities involved.

*Suppose that there are 32 biscuits to be shared among 8 people. How can the numbers in the problem be altered without affecting the size of the share?*




Halving both the number of biscuits and the number of people who share them keeps the size of the shares the same: 32 ÷ 8 = 4, 16 ÷ 4 = 4, and 8 ÷ 2 = 4. The same is true no matter what number the numbers of biscuits and people are divided by. The size of the shares is also unaffected if the numbers of biscuits and people are doubled, trebled, or multiplied by any number at all, although this operation will seldom make a division problem easier. Algebraically, the principle can be expressed in this way: if $a \div b = c$, then $\frac{a}{n} \div \frac{b}{n} = c$ where $n$ is any number except zero.

Students will be able to use this method as long as they can spot a value for $n$ that works, that is, a value that is a common factor of both numbers. For example, in 198 ÷ 6 = □, students won't be able to reduce the problem to 99 ÷ 3 = □ unless they realise that both 198 and 6 are even numbers and so have a common factor of 2.

Sometimes, it will take too much work to simplify a division problem using this method and another strategy is more suitable. Students need to recognise when halving, thirding, and so on of both numbers is an effective strategy and when it is not. Effectiveness depends on how easy it is to find a common factor and how easy it is to divide by that common factor. Take, for example, 657 ÷ 9 = □. Both 657 and 9 are divisible by 3, but the difficulty of dividing 657 by 3 means that, for many students, standard place value methods are likely to be a better choice. For example: 630 ÷ 9 = 70; 27 ÷ 9 = 3; 70 + 3 = 73.

In question **1**, the unknown is the total number of biscuits on the table; in question **2**, it is the size of each share. It is important that students meet division problems posed in different ways like this so that they come to understand the principle of inverse or reversibility.

Students need to be able to process question **3** without first solving the problem posed at the beginning: 168 ÷ 24 = □. That is, they need to work backwards through a number of equivalent statements: 168 ÷ 24 = ☆ ÷ 12 = ◇ ÷ 6 = ○ ÷ 4 = △ ÷ 3. In this way, they operate on statements of equality while accepting the lack of closure that comes from not knowing the quotient (answer). They know that ☆ must be half of 168 because 12 is half of 24, and so on. This is the same kind of thinking that underpins equivalent fractions, for example, $\frac{168}{24} = \frac{☆}{12}$.

The problems in question **4** can be solved in similar ways, but because the numbers are deliberately larger, the students will have to find the factors of the divisors (36, 28, and 27) before they can work backwards.

For question **5**, the students should be very sure that their equations can be sensibly solved by strategies such as halving and doubling, place value, or working backwards, before they try them out on a classmate.

**Links**

**Numeracy Project materials** (see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 6: Teaching Multiplication and Division*
  Royal Cooking Lesson, page 30

**Figure It Out**

- *Number Sense and Algebraic Thinking: Book Two,* Level 3
  Horsing Around, page 11

# Pages 8–9: Leftovers

**Achievement Objectives**

- write and solve problems involving decimal multiplication and division (Number, level 4)
- demonstrate knowledge of the basic units of length, mass, area, volume (capacity), and temperature by making reasonable estimates (Measurement, level 3)

| |
|---|
| AC |
| EA |
| AA |
| AM |
| AP |

**Number Framework Links**

Use this activity with stage 7 students to encourage them to think about the meaning of the fractional parts of quotients and to interpret them sensibly in the context.

## Activity

This activity explores how best to express remainders in division problems. Suitable expression depends on the context of the problem, so sometimes the decimal given by a calculator calculation will make sense and sometimes it will not.

Most of the contexts in Leftovers involve division of measurements. For example, the milk quantity is measured in litres, the foodwrap is measured in metres, and the time in hours and minutes.

In most of these cases, the decimal produced in a calculator division will have meaning in terms of the measurement units. 7 kilograms of potatoes shared among 8 people can be calculated as 7 ÷ 8 = 0.875 kg. This is equivalent to 875 grams by conversion of units because 1 kilogram is equivalent to 1 000 grams. The synergy between the metric system of measures and our base 10 number system makes this possible. In measurement contexts such as time, where the base may be 60 or 12, this synergy does not exist.

In many contexts, however, the exact (calculator) answer is not appropriate. For example, for the potatoes in this activity, the best solution would be to get each share as close as possible to 875 grams without cutting potatoes into pieces. For the flying fox time, students need to realise that 2 hours and 45 minutes is not the same as 2.45 hours. 45 minutes is $\frac{45}{60}$ of an hour, not $\frac{45}{100}$, because there are 60 minutes in an hour. The best approach in this case is to convert 2 hours and 45 minutes into minutes (2 x 60 + 45 = 165 minutes). The calculator now gives 165 ÷ 8 = 20.625 minutes.

Most students are likely to have trouble interpreting 0.625 of a minute because it does not mean 62.5 seconds! In the context, 20 minutes is an acceptable answer to the question. No matter how much the campers value their turn on the flying fox, they could not or would not attempt to equalise time to the nearest second. Encourage your students to discuss the realities of the situation.

The sharing of discrete (separate) objects can present a different problem because it is often inappropriate to divide separate, whole objects into fractions, especially if the objects are living creatures! Biscuits, however, can be partitioned with minimal social consequences. The calculator operation gives 5 x 30 ÷ 8 = 18.75. A decimal to fraction conversion is needed: 0.75 = $\frac{3}{4}$. So each group gets $18\frac{3}{4}$ biscuits. Practically, this is not likely to be a sensible outcome, so it might be that 6 groups get 19 biscuits each and 2 groups get just 18. Let your students discuss how they would prefer to see this sharing problem resolved.

In question **3** in **Activity Two**, students generalise about the situations when a calculator division output is helpful.

From their investigations, they should conclude that the calculator answer is always helpful when the problems involve some kind of metric measurement but that the practicalities of the situation often mean that some rounding is required. When discrete objects are shared, it is only the decimal part of the quotient that needs interpretation. This usually involves either converting the decimal part to a tidy common fraction or concluding that the objects should not be partitioned further and giving a whole-number answer.

**Links**

**Numeracy Project materials** (see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 6: Teaching Multiplication and Division*
  Remainders, page 32

**Figure It Out**

- *Basic Facts*, Level 3
  It Remains to Be Seen, page 22
- *Number*: *Book Three,* Years 7–8, Level 4
  Digit Challenge, page 18
  Team Leaders, page 10
- *Number*: *Book Five,* Years 7–8, Level 4
  Revisiting Remainders, page 1
  Remainder Bingo, page 2
- *Number Sense and Algebraic Thinking*: *Book One,* Level 3
  Just Right!, pages 8–9
- *Number Sense and Algebraic Thinking*: *Book Two,* Level 3
  Triple Trouble, page 1

## Pages 10–11: The Power of 10

**Achievement Objectives**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- write and solve problems involving decimal multiplication and division (Number, level 4)

AC
EA
AA
AM
AP

**Number Framework Links**

Students who are using advanced multiplicative strategies (stage 7) are likely to benefit most from this activity.

### Activity

The questions in this activity are about multiplication and division by powers of 10. Powers of 10 are created by multiplying tens together. Some powers of 10 are:
10 x 10 = 100 (ten tens equal one hundred)
10 x 10 x 10 = 1 000 (ten times ten times ten equals one thousand)
10 x 10 x 10 x 10 = 10 000 (ten times ten times ten times ten equals ten thousand)
10 x 10 x 10 x 10 x 10 = 100 000 (ten times ten times ten times ten times ten equals one hundred thousand).

Powers of 10 can be written using exponents, for example, $10^3$ = 1 000. The "3" indicates that 3 tens have been multiplied together. There are also 3 zeros in the product (1 000). Powers of 10 with an exponent of 1 or greater are counting numbers {1, 2, 3, 4, ...}. Powers of 10 may also be less than 1, but their meaning will be less obvious. Students will find it helpful to create this pattern for exponents greater than or equal to 1 and then extend it to the left:

| $10^{-3}$ | $10^{-2}$ | $10^{-1}$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ |
|---|---|---|---|---|---|---|
| 0.001 | 0.01 | 0.1 | 1 | 10 | 100 | 1000 |
| One-thousandth | One-hundredth | One-tenth | One | Ten | One hundred | One thousand |

Powers of 10 are created by multiplication by 10, so moving one column to the left in the table above equates to division by 10. For example, $\frac{1}{10}$ of 100 is 10, so $\frac{1}{10}$ of $\frac{1}{10}$ is $\frac{1}{100}$. Pages 22–27 of *Book 7: Teaching Fractions, Decimals, and Percentages* from the NDP resources describe how materials such as deci-mats or decimal pipes can be used to model these patterns in the place value system.

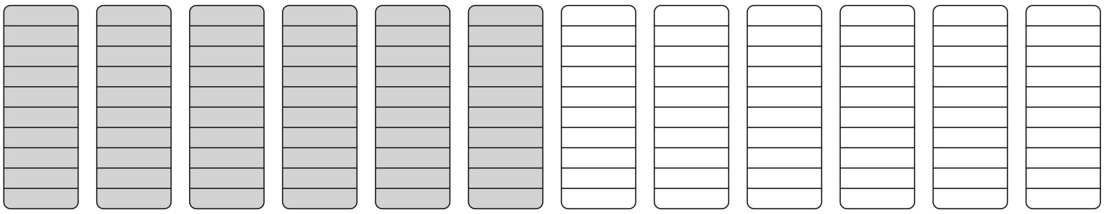
Students must be able to understand multiplication and division by powers of 10 if they are to handle more complex problems. Some may still think that "multiplication makes bigger" and "division makes smaller". This overgeneralisation is based on what happens with whole numbers. As the examples in the chart below illustrate, the opposite is true in many cases. You need to work with your students to correct this common error of reasoning.

| **Whole numbers** | **Decimals less than 1 but greater than 0** |
|---|---|
| 5 x 4 = 20 (answer is greater than 4) | 0.5 x 4 = 2 (answer is less than 4) |
| 20 ÷ 5 = 4 (answer is less than 20) | 2 ÷ 0.5 = 4 (answer is greater than 2) |

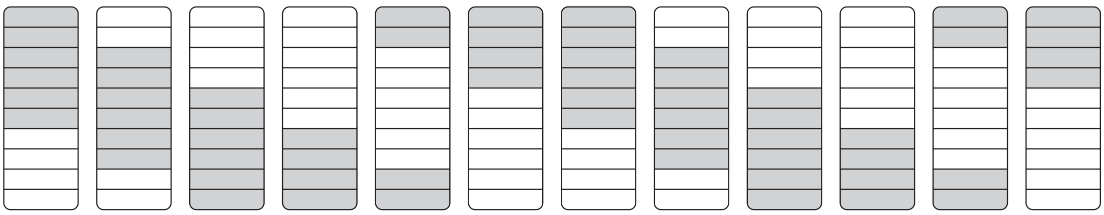
The questions in this activity use a measurement context for division rather than an equal-sharing context. Although it is reasonable to consider 12 objects measured in lots of 0.6, it is quite a stretch to imagine 12 objects shared into 0.6 equal sets. Problems that involve metric measurement are good vehicles for developing division by decimals.

Once the students have answered question **1**, it is important that they discuss what they have found and can see the generalisation that their answer points to: *If you measure the same amount in units that are 10 times smaller, the number of units that can be fitted in will be 10 times greater.* The following two diagrams illustrate this mathematical principle.

12 ÷ 6 = 2 because 2 measures of 6 litres go into 12 litres.



12 ÷ 0.6 = 20 because 20 measures of 600 millilitres (0.6 litres) go into 12 litres. The measures are 10 times smaller, so 10 times as many measures fit into the whole (12 litres).



Similarly, if you multiply a number by a number that is 10 times greater than the one you used before, the result will be 10 times greater. For example, 3 x 7 = 21, so 30 x 7 = 210. This means also that 0.3 x 7 = 2.1, because 0.3 is 10 times smaller than 3.

The parts of questions **3** and **5** can all be solved by multiplying or dividing the known result by 10 or a power of 10. For example, 6 ÷ 3 = 2 has an answer that is 10 times smaller than the answer to 60 ÷ 3 = 20 or 10 times greater than the answer to 0.6 ÷ 3 = 0.2.

In some problems, both the starting number and the divisor have been changed by factors of 10, 100, 1 000, and so on. For example, in question **3b**, 2 ÷ 0.4 = ☐ can be easily solved using the known relationship 20 ÷ 4 = 5: the starting number 2 is 10 times smaller than 20, and 0.4 is 10 times smaller than 4, so the net effect is that the answer will be the same: 20 ÷ 4 = 5 and 2 ÷ 0.4 = 5.

In question **4**, the students make generalisations about the effect of the size of the divisor on the results of divisions. Here are three very useful generalisations that apply to all *positive numbers* (things get more complicated when zero or negative numbers are involved):

- Dividing by a number greater than 1 will give a result that is less than the starting number. (For $a \div b = c$, if $b > 1$, then $c < a$.)
- Dividing a number by itself always gives 1. (For $a \div b = c$, if $a = b$, then $c = 1$.)
- Dividing by a number that is less than 1 but greater than 0 will give a result that is greater than the starting number. (For $a \div b = c$, if $0 < b < 1$, then $c > a$.)

Two similar but different generalisations apply to multiplication of *positive numbers*:

- Multiplying by a number greater than 1 will give a result that is greater than the starting number. (For $a \times b = c$, if $b > 1$, then $c > a$.)
- Multiplying by a number that is less than 1 but greater than 0 will give a result that is less than the starting number. (For $a \times b = c$, if $0 < b < 1$, then $c < a$.)

Encourage your students to discover these generalisations for themselves and to express them in their own words. They should generalise these properties across a range of numbers, including whole numbers and decimals. Using calculators can provide answers without the burden of calculation and allows the students to focus on the underpinning number relationships.

**Links**

**Numeracy Project materials** (see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Folding Fractions and Decimals, page 36
  Pipe Music with Decimals, page 22
  Deci-mats, page 25
- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Estimation in Decimal Multiplication & Division Problems, page 25
  Multiplication of Decimal Fractions, page 37

**Figure It Out**

- *Number: Book Two,* Levels 3–4
  Spring Fever, page 6
  Ageing in Space, page 8
  Meal Deal, page 9
- *Number: Book Three,* Levels 3–4
  Dog's Dinner, page 14
- *Number Sense and Algebraic Thinking: Book Two,* Levels 3–4
  Using Mates, pages 16–17
  Compatible Multiples, page 21
- *Number Sense: Book Two,* Years 7–8, Level 4
  Astronomical Proportions, pages 16–17
  Line Up, page 20
- *Number: Book Five,* Years 7–8, Level 4
  Body Mass, page 10
- *Number: Book Six,* Years 7–8, Level 4+
  Accident-prone, page 11

# Pages 12–13: Spoilt for Choice

**Achievement Objective**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)

| |
|---|
| AC |
| EA |
| AA |
| AM |
| AP |

**Number Framework Links**

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

## Activity

The questions in this activity explore multiplication as a Cartesian product (cross product), named for the famous French mathematician Rene Descartes, who invented the number plane. The Cartesian product gives the number of different combinations to be found where items from two different sets are matched one to one.

You could introduce this concept to your students by asking them to imagine someone matching 4 different tops (red, blue, yellow, and green) with 3 different pairs of shorts (red, blue, and yellow). How many different combinations could they make?

The possibilities could be modelled with cubes by organising the members of each set in a table or matrix:

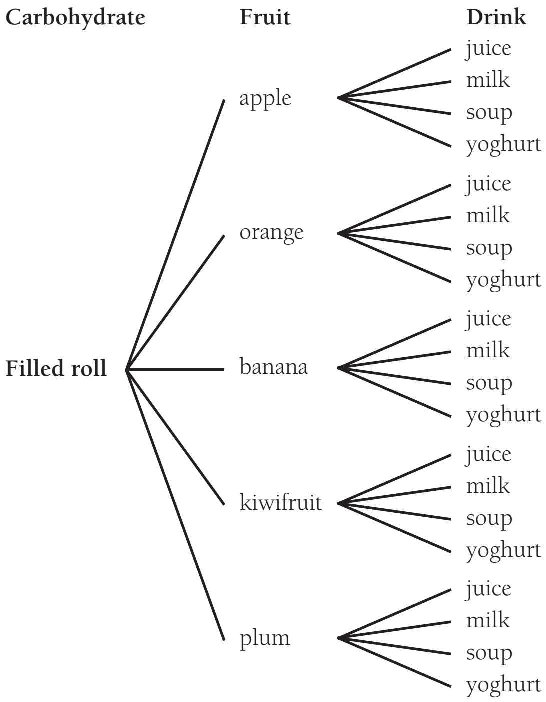
**Tops**

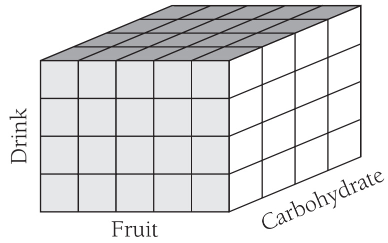
| | | Red | Blue | Yellow | Green |
|---|---|---|---|---|---|
| **Shorts** | Red | R<br>R | B<br>R | Y<br>R | G<br>R |
| | Blue | R<br>B | B<br>B | Y<br>B | G<br>B |
| | Yellow | R<br>Y | B<br>Y | Y<br>Y | G<br>Y |

The table shows that 4 x 3 = 12 different combinations are possible. Next, have the students imagine that the person acquires another pair of shorts. Ask *How many combinations are now possible?* (4 x 4 = 16). So in general, if *n* objects are matched in one-to-one correspondence with *r* objects, there are *n* x *r* possible combinations.

In Spoilt for Choice, the questions involve combinations of items from three sets, not two as in the example above. The combinations in this case can be shown on a tree diagram, viewed as a 3-dimensional space, or represented in a chart like the one above but with the third set of items included in each cell. Each of these three representations could be introduced using a simplified problem: one in which there are fewer choices (say 2, 2, and 2).

A complete tree diagram showing 4 x 5 x 4 = 80 combinations would be large but can be drawn easily using a computer drawing program and grouping, copying, and pasting repeated items. The partial tree diagram below shows the combinations of fruit and drink that could accompany a filled roll. There are 20 such combinations, and all 20 could equally well accompany any of the 4 carbohydrate options, so that is 4 x 20 = 80 combinations in all.




A cuboid can be used to represent the combinations. Each of the small cubes represents a particular combination of the three items. The number of small cubes is 4 x 5 x 4 = 80. As the cuboid shows clearly, the number of combinations for *n*, *r*, and *p* objects matched in one-to-one correspondence is $n \times r \times p$.

The diagram in the answers for question **1a** shows how a 2-dimensional chart can be adapted to include a third set of options.

The number of combinations for the new menu is 4 x 5 x 4 = 80. 80 ÷ 5 = 16 school weeks, which is more than 1 term, so the tuck shop's claim is true.

Question **3** requires the students to anticipate the effect of adding one more item to the menu. If the item is another fruit, then the number of different combinations will be 4 x 6 x 4, which is 4 x 1 x 4 = 16 more than previously. Adding another carbohydrate or drink will produce 5 x 5 x 4 or 4 x 5 x 5 combinations, which is 4 x 5 x 1 = 20 more than previously. Given that there are only two possibilities (add a fruit or add a carbohydrate or drink), the students can quickly compare the effect of each.

Question **4** tests students' understanding of the ideas developed in the earlier questions. If it is assumed that there is a choice of items in each category, there must be a minimum of 2 in each list: 2 x 3 x 3 x 3 = 54, 2 x 2 x 3 x 5 = 60, and 2 x 2 x 4 x 4 = 64 are three possible answers.

Question **5** is suitable for pair or group discussion. Allowing various combinations of fillings greatly increases the number of choices. For example, suppose there were only three options: ham, chicken, cheese. With these choices alone, the filling combinations could be: ham (by itself); cheese (by itself); chicken (by itself); ham and chicken; ham and cheese; chicken and cheese; and ham, chicken, and cheese – that's 7 choices. If a fourth option is added, there are 15 choices.

## Pages 14–15: Primes and Emirps

**Achievement Objective**

- classify objects, numbers, and ideas (Mathematical Processes, developing logic and reasoning, levels 3–4)

AC
EA
AA
AM
AP

**Number Framework Links**

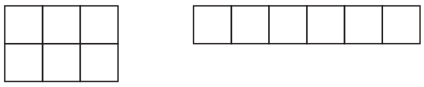
Students will need to be using at least advanced multiplicative strategies (stage 7) to solve these problems.

### Activities and Investigations

These activities and investigations are based around prime and non-prime numbers, including square numbers.

**Activity One** begins with an illustration that makes a connection between the number of factors that a number has and the number of rectangles that can be formed with that many square tiles.

Take, for example, the number 6. Two rectangles can be formed from 6 tiles:



The dimensions of these rectangles, 2 x 3 and 1 x 6, correspond to the factors of 6, which are {1, 2, 3, 6}.

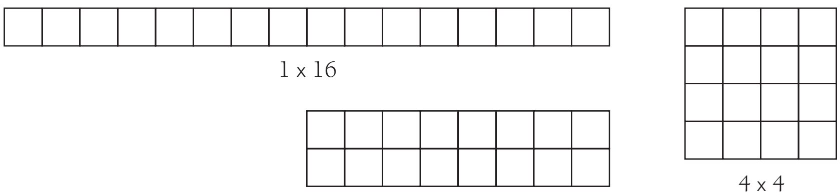
For each prime number, only one rectangle can be formed. For example, here are the rectangles for the prime numbers 7 and 13:

1 x 7 1 x 13

So the factors of 7 are {1, 7} and the factors of 13 are {1, 13}. It is also true to say that, in the set of whole numbers, 7 is only divisible by 1 and 7 and 13 is only divisible by 1 and 13.

Prime numbers are sometimes referred to as non-rectangular numbers, and numbers with 3 or more factors as rectangular numbers.

As can be seen with the factors of 6, factors usually exist in pairs. The number 12 has these pairs: 1 x 12, 2 x 6, and 3 x 4. Most rectangular numbers have an even number of factors – but not all do. For example, 16 is a rectangular number, but it has an odd number of factors: {1, 2, 4, 8, 16}.



1 x 16 2 x 8 4 x 4

The reason for this is that 16 is a square number, which is a special kind of rectangular number. A square number has a factor that is used twice (in the case of 16, this factor is 4), but it only appears once on the list of factors for that number.

Rectangular numbers are also referred to as composite numbers because they are composed of prime factors. This means that they can be expressed as the product of primes. For example, 12 = 2 x 2 x 3 and 15 = 3 x 5. The prime factorisation of any composite number is unique; no other combination will work. This is known as "the fundamental theorem of arithmetic".

This theorem is the basis of the sieve of Eratosthenes, a method for finding prime numbers, which is explored in **Activity Two**. Eratosthenes, a mathematician of ancient Greece, was famous for calculating the circumference of the Earth. His sieve works by eliminating all the composite numbers, leaving only the prime numbers behind.

The process involves removing the multiples of 2, then 3, then 5, then 7, and so on. Because these numbers are multiples, they must be composite. The sieving process only needs to be continued until the square root of the largest number in the sieve is reached. So for the numbers 1 to 100, the multiples of 7 are the last to be eliminated because the next prime (11) is greater than the square root of 100 (10). This works because factors exist in pairs and for every prime factor less than the square root, there is a complementary factor greater than the square root. The factors close in on the square root from below and above.

**Investigation One** asks students to work out a way to find primes up to 200. The Answers explain how to do this. The closest square root to 200 is $\sqrt{196} = 14$, so the multiples of 13 are the last that need to be found. You could extend this activity by asking your students to find the primes up to 900. To do this, all the multiples of 2, 3, 5, 7, 11, 13, … 29 must be eliminated because these numbers are the primes less than 30, which is the square root of 900.

Prime numbers up to 10 000 can be found in this way using a spreadsheet program. The square root of 10 000 is 100, so each number needs to be divisibility-tested only for prime numbers up to and including 97.

The spreadsheet shown here can be used to test any number up to 10 000 and see if it is prime. The number to be tested is entered in cell D2.

To set up this spreadsheet, enter all the prime numbers up to 97 in column A. In cell B2, enter the formula =$D$2/A2, then fill down in column B to cell B26 (opposite 97 in column A). The formula checks the number entered in D2 for divisibility by each of the prime numbers in column A. If a whole number appears in column B, then the number in cell D2 is divisible by the complementary prime in column A, so it cannot be prime itself. In the example illustrated, 1 791 can't be prime because 3 goes into it 597 times.

=$D$2/A2

| | A | B | C | D |
|---|---|---|---|---|
| 1 | Prime | Quotient | | Test |
| 2 | 2 | 895.5 | | 1791 |
| 3 | 3 | 597 | | |
| 4 | 5 | 358.2 | | |
| 5 | 7 | 255.857143 | | |
| 6 | 11 | 162.818182 | | |
| 7 | 13 | 137.769231 | | |
| 8 | 17 | 105.352941 | | |
| 9 | 19 | 94.2631579 | | |
| 10 | 23 | 77.8695652 | | |
| 11 | 29 | 61.7586207 | | |
| 12 | 31 | 57.7741935 | | |
| 13 | 37 | 48.4054054 | | |
| 14 | 41 | 43.6829268 | | |
| 15 | 43 | 41.6511628 | | |
| 16 | 47 | 38.106383 | | |
| 17 | 53 | 33.7924528 | | |
| 18 | 59 | 30.3559322 | | |
| 19 | 61 | 29.3606557 | | |
| 20 | 67 | 26.7313433 | | |
| 21 | 71 | 25.2253521 | | |
| 22 | 73 | 24.5342466 | | |
| 23 | 79 | 22.6708861 | | |
| 24 | 83 | 21.5783133 | | |
| 25 | 89 | 20.1235955 | | |
| 26 | 97 | 18.4639175 | | |
| 27 | | | | |

### Investigation Two

The students can use the primes they found in **Investigation One** to find some lower value emirps in question **2**. Have them discuss how they approached finding other 2- and 3-digit emirps. An Internet search on emirps, as suggested in the Answers, would be worthwhile, and the students could extend that to searching for information on primes. If nothing else, they will be exposed to the passion for maths that many mathematicians feel.

**Links**

Teaching ideas about prime factors and prime factorisation can be referenced through the Algebra planning units for stage 6 on www.nzmaths.co.nz/numeracy/PlanLinks/default.aspx

**Figure It Out**

- *Number: Book Four,* Years 7–8, Level 4
  In Your Prime, page 1
  Going Bananas, page 2
  Going to Extraordinary Lengths, page 4
  Boxing Balls, page 5
  Prime Sites, page 6
  Igloo Iceblocks, page 7
- *Number: Book Six,* Years 7–8, Level 4+
  Digital Delights, page 2
  Factor Towers, page 7

## Pages 16–18: Prime Time

**Achievement Objective**

- classify objects, numbers, and ideas (Mathematical Processes, developing logic and reasoning, levels 3–4)

AC
EA
AA
AM
AP

**Number Framework Links**

Use Activity One to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7). Students will need to be at least at stage 7 (advanced multiplicative) to solve the rest of the problems.

### Activities One to Three

These activities follow Primes and Emirps and build on the activities and investigations that the students have already done on those pages.

When the students are working through these activities, it is important that they understand the difference between "find the factors of", "find the prime factors of", and "factorise".

In the first two instances, the answer will be a list of numbers, often (though not necessarily) expressed as a set and using set brackets. For example, the factors of 80 are {1, 2, 4, 5, 8, 10, 16, 20, 40, 80}. However, only two of these factors are primes, so the prime factors of 80 are {2, 5}.

In the third instance, the answer will be expressed as two or more numbers multiplied together. Often, this means "express as the product of its primes", for example, 80 = 2 x 2 x 2 x 2 x 5. It can also mean "express as the product of two factors", for example, 80 = 8 x 10.

In **Activity One**, the students need to find all the factors of 36, 56, and 84. They can do this without delving into primes, but if they begin by finding the prime factors, they will find it very easy to find all the factors of that number.

Before the students attempt the questions on the page, work with them on finding the factors of some other numbers. For example, 70 = 2 x 5 x 7. Obviously 1, 2, 5, 7, and 70 are factors of 70, but so also are 10, 14, and 35. These factors are found by pairing the primes, for example, 2 x 7. Students need to consider all possible pairings to be sure that they have found all possible factorisations.

Question **3** is a little more difficult in that the students need to consider combinations of *three* factors that multiply to give 84. As before, the first step is to establish that the prime factors of 84 are {2, 2, 3, 7}, so 84 = 2 x 2 x 3 x 7. The next step is to systematically look for combinations of 3 factors that will multiply to give 84:

- with 1 as the smallest factor: 1 x 1 x 84, 1 x 2 x 42, 1 x 3 x 28, 1 x 4 x 21, 1 x 6 x 14, 1 x 7 x 12
- with 2 as the smallest factor: 2 x 2 x 21, 2 x 3 x 14, 2 x 7 x 6 (no 1s allowed as these are considered in the first case above)
- with 3 as the smallest factor: 3 x 4 x 7 (no 1s or 2s allowed as these have been considered in the two cases above).

Prime factorisations make it easy to find common factors for two or more numbers. For example, 36 = 2 x 2 x 3 x 3 and 27 = 3 x 3 x 3. Their prime factors indicate that they have a common factor of 3 x 3 = 9. This means that $\frac{27}{36}$ can be expressed as $\frac{3 \times \cancel{(3 \times 3)}}{2 \times 2 \times \cancel{(3 \times 3)}} = \frac{3}{2 \times 2} = \frac{3}{4}$. (What we have done here is take out a common factor of 9, commonly called cancelling.)

In **Activity Three**, question **2**, the students use this idea to solve problems involving rates. Expressed as a product of prime factors, 210 = 2 x 3 x 5 x 7. Similarly, 42 = 2 x 3 x 7. The prime factors 2, 3, and 7 are common to both, so the factor 2 x 3 x 7 must also be common to both. So 210 ÷ 42 = (2 x 3 x 5 x 7) ÷ (2 x 3 x 7), which equals 5.

**Activity Four** extends the ideas of divisibility to decimals that either terminate (end) or do not terminate. For any division 1 ÷ *n* = ☐, the resulting decimal will terminate if *n* divides evenly into 10, 100, 1 000, and so on, because 1 is 10 tenths, 100 hundredths, 1 000 thousandths, and so on. Students who investigate this will find that this is only true if the prime factors of *n* are a combination of 2 and/or 5. For example, 1 ÷ 64 = 0.015625. This decimal terminates because 64 = 2 x 2 x 2 x 2 x 2 x 2 = $2^6$, so 64 divides evenly into one-million-millionths (another name for 1).

1 ÷ 36 = $0.02\overline{77}$ does not terminate because the prime factors of 36 are 2 x 2 x 3 x 3 and 9 does not divide evenly into any power of 10.

See pages 38–39, *Book 7: Teaching Number Sense and Algebraic Thinking* for more on these ideas.

### Challenge

Students should generate a list of fractions in which the numerator is not 1 and then use a calculator to find which of them terminate. For ease of investigation, suggest that they limit their examples to fractions that have numbers no greater than 100 for both the numerator and denominator. Once they have found which of the fractions on their list terminate, they should factorise the numerators and denominators, where possible, and then look for common elements. They will find that as long as the only prime factors of both numerator and denominator are 2 and/or 5, the decimal will terminate. If another prime is involved, the decimal will terminate only if that prime is a factor of both the numerator and the denominator. For example, $\frac{28}{175}$ terminates because $\frac{28}{175} = \frac{2 \times 2 \times 7}{5 \times 5 \times 7}$ and 7 is a factor of both the numerator and the denominator.

**Links**

Teaching ideas about prime factors and prime factorisation can be referenced through the Algebra planning units for stage 6 on www.nzmaths.co.nz/numeracy/PlanLinks/default.aspx

**Figure It Out**

- *Number: Book Four,* Years 7–8, Level 4
  In Your Prime, page 1
  Going Bananas, page 2
  Going to Extraordinary Lengths, page 4
  Boxing Balls, page 5
  Prime Sites, page 6
  Igloo Iceblocks, page 7

- *Number: Book Six*, Years 7–8, Level 4+
  Digital Delights, page 2
  Factor Towers, page 7

## Page 19: Fred's Rent-a-Fence

**Achievement Objectives**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- calculate perimeters of circles, rectangles, and triangles, areas of rectangles, and volumes of cuboids from measurements of length (Measurement, level 4)

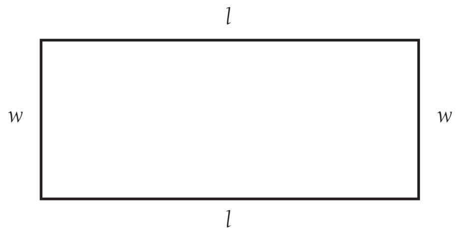
AC EA AA AM AP

**Number Framework Links**

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

### Activity

This activity is based on the perimeter and area of rectangles. As a general introduction, have your students look at this rectangle where the side lengths are given as $l$ and $w$.




The area can be found by: area = $l \times w$.
The perimeter can be found by: perimeter = $2 \times l + 2 \times w$.

Problems that involve maximising or minimising one measurement while either holding the other constant or minimising it are common in the real world. Fred's fence is typical of constrained maximisation or minimisation problems.

Students exploring question **1** are likely to try different side lengths that will result in an area of 80 square metres. The problem requires a systematic approach, so encourage your students to organise their results in a table or organised list:

| Side A | Side B | Area | Perimeter |
|---|---|---|---|
| 1 | 80 | 80 | 162 |
| 2 | 40 | 80 | 84 |
| 4 | 20 | 80 | 48 |
| 5 | 16 | 80 | 42 |
| 8 | 10 | 80 | 36 |

In this way, the students can find all the solutions with whole-number side measurements and calculate the perimeters at the same time. They may notice that the closer the side measurements become to each other, the smaller the perimeter becomes.

Encourage your students to explore the minimum perimeters for rectangles with the areas 16, 36, and 64 (square numbers). They will find that the perimeter is minimised when the rectangle is a square. In this situation, the length of each side is the square root of the area. They can then go back to question **1** with the knowledge that the solution is the closest whole number to $\sqrt{80}$ = 8.944 (to 4 significant figures). Students are likely to argue that the question asked for a rectangle and that a rectangle is not a square. It is worth stopping to discuss this reasonable view. In everyday use, a rectangle and a square *are* different shapes, but in mathematics, a square is just a special case of a rectangle.

Provide the students with a set of rectangles and squares and ask them to describe the attributes of these shapes. Encourage them to come up with minimal definitions, listing just the attributes that are absolutely necessary to define the shape. Students will typically say that a rectangle has:

- 4 sides
- 4 right-angled corners
- 2 pairs of parallel sides.

If you ask them to draw a 4-sided polygon that has right-angled corners but does not have 2 pairs of parallel sides, they will find that this is impossible. So it is not necessary to state that opposite sides must be parallel. This gives us the minimal definition for a rectangle. The minimal definition of a square is "a 4-sided polygon with right-angled corners and equal sides". Squares are therefore a subclass of rectangles.

In the **Investigation**, students try to find rectangles that have the same number for the measurement of their perimeter as they do for the measurement of their area.

One solution is a square with sides of 4 metres. Its perimeter is 16 metres, and its area is 16 square metres. If they are systematic, students should be able to establish the existence of two other whole-number solutions.

They could begin by setting the length (at, say, 2 metres) and exploring what widths might work. They will discover that no whole-number solution will work for a side length of 2. But if they then try 3, they will find that a 3 x 6 rectangle has an area of 18 square metres and a perimeter of 18 metres. 6 x 3 is a third solution, but this is not a genuinely different rectangle.

Having got this far, your students may guess that there are other rectangles that meet the requirement but that they do not have whole-number sides. There are in fact an infinite number of such rectangles. In the table below, there are six rectangles that happen to have a whole-number measurement for one of their two dimensions. You could give your students the length of side *b* and challenge them to find the length of side *a* (in bold in the table), using a trial-and-improvement strategy.

| **Side *a*** | **Side *b*** | **Area** | **Perimeter** |
|---|---|---|---|
| 7 | 2.8 | 19.6 | 19.6 |
| **10** | 2.5 | 25 | 25 |
| **12** | 2.4 | 28.8 | 28.8 |
| **18** | 2.25 | 40.5 | 40.5 |
| **22** | 2.2 | 48.4 | 48.4 |
| **27** | 2.16 | 58.32 | 58.32 |

There is an algebraic relationship between the pairs of values of *a* and *b* that satisfy the requirement that the number of perimeter units must be equal to the number of units of area. The relationship can be expressed in this way: $b = \frac{2a}{a-2}$. (To find the length of the second side, double the length of the first and divide by its length less 2.) Students who are developing an understanding of symbolic notation may like to try using this formula to find other pairs for *a* and *b* with the help of a calculator or spreadsheet program such as that shown.

=2*A2/(A2-2)

| | A | B | C | D |
|---|---|---|---|---|
| 1 | a | b | perimeter | area |
| 2 | 3 | 6 | 18 | 18 |
| 3 | 4 | 4 | 16 | 16 |
| 4 | 5 | 3.33333333 | 16.6666667 | 16.6666667 |
| 5 | 6 | 3 | 18 | 18 |
| 6 | 7 | 2.8 | 19.6 | 19.6 |
| 7 | 8 | 2.66666667 | 21.3333333 | 21.3333333 |
| 8 | 9 | 2.57142857 | 23.1428571 | 23.1428571 |
| 9 | 10 | 2.5 | 25 | 25 |
| 10 | 11 | 2.44444444 | 26.8888889 | 26.8888889 |
| 11 | 12 | 2.4 | 28.8 | 28.8 |
| 12 | 13 | 2.36363636 | 30.7272727 | 30.7272727 |
| 13 | 14 | 2.33333333 | 32.6666667 | 32.6666667 |
| 14 | 15 | 2.30769231 | 34.6153846 | 34.6153846 |
| 15 | 16 | 2.28571429 | 36.5714286 | 36.5714286 |
| 16 | 17 | 2.26666667 | 38.5333333 | 38.5333333 |
| 17 | 18 | 2.25 | 40.5 | 40.5 |
| 18 | 19 | 2.23529412 | 42.4705882 | 42.4705882 |
| 19 | 20 | 2.22222222 | 44.4444444 | 44.4444444 |
| 20 | 21 | 2.21052632 | 46.4210526 | 46.4210526 |
| 21 | 22 | 2.2 | 48.4 | 48.4 |
| 22 | 23 | 2.19047619 | 50.3809524 | 50.3809524 |
| 23 | 24 | 2.18181818 | 52.3636364 | 52.3636364 |
| 24 | 25 | 2.17391304 | 54.3478261 | 54.3478261 |
| 25 | 26 | 2.16666667 | 56.3333333 | 56.3333333 |
| 26 | 27 | 2.16 | 58.32 | 58.32 |
| 27 | 28 | 2.15384615 | 60.3076923 | 60.3076923 |
| 28 | 29 | 2.14814815 | 62.2962963 | 62.2962963 |
| 29 | 30 | 2.14285714 | 64.2857143 | 64.2857143 |

**Links**

**Numeracy Project materials**

(see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 9: Teaching Number through Measurement, Geometry, Algebra and Statistics* Investigating Area, page 11

**Figure It Out**

- *Number: Book Three,* Years 7–8, Level 4 Orchard Antics, page 23
- *Number Sense and Algebraic Thinking: Book One,* Levels 3–4 Tile the Town, Tiny!, pages 20–21

## Pages 20–21: Dotty People

**Achievement Objectives**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- write and solve problems involving decimal multiplication and division (Number, level 4)

AC
EA
AA
AM
AP

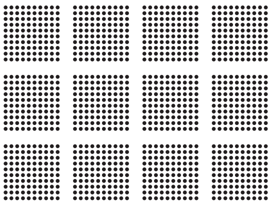
**Number Framework Links**

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

### Activity

This activity explores multiplicative place value, a concept essential to students' understanding of large whole numbers and their ability to perform multiplicative calculations. The activity is based around the use of a 10 by 10 dot array. Students know that this array contains 100 dots, so it provides a good starting point for further exploration.

It is essential that, when calculating with large whole numbers and decimals, students pay attention to unit size. Rules that they may have learned, such as "add a zero" to multiply by 10, work with the simple examples that they first encounter but lead to serious misconceptions when working with numbers that have a decimal point, for example, the misconception that 2.4 x 10 = 2.40. When multiplying by 10, the students must make the connection between a basic fact and the units being multiplied. For example, we can represent 40 x 30 = ☐ with dots, as follows:



The display shows that 4 x 3 = 12 is embedded in the array. The 12 units are each one hundred (10 x 10). Extending that idea gives results such as:
300 x 40 = 12 000 because the units are 100 x 10 = 1 000 (one hundred tens is 1 000)
0.3 x 400 = 120 because the units are 0.1 x 100 = 10 (one-tenth of 100 is 10)
0.3 x 0.4 = 0.12 because the units are 0.1 x 0.1 = 0.01 (one-tenth of one-tenth is one-hundredth).

Solving problems of this type requires a number of key understandings:

- Multiplication means "of". Just as 7 x 4 means 7 sets of 4, $\frac{1}{2} \times \frac{1}{2}$ means one-half of one-half and $\frac{1}{4}$ x 20 means one-quarter of 20.
- Multiplication does not always result in an answer that is bigger than either of the original two numbers. Students generalise incorrectly from whole-number examples such as 3 x 5 = 15 that "multiplication makes bigger". Contradictions emerge when they encounter examples such as $\frac{1}{3} \times 6 = 2$ or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
- The base 10 place values involve a pattern of multiplying by 10 (a movement to the left), for example, 10 x 100 is 1 000, and dividing by 10 (a movement to the right), for example, 1 divided by 10 is one-tenth. In this case, the division means dividing into 10 equal parts, for example, one-hundredth divided into 10 equal parts gives one-thousandths.
- Equivalent fractions are essential to understanding the place value units involved in decimal multiplications. For example, one-tenth of one-tenth equals one-hundredth (0.1 x 0.1 = 0.01); this is derived from the fact that $\frac{1}{10}$ is equivalent to 10 hundredths.

While they are working through the problems in Dotty People, ask the students to find the sides of the rectangles for which they are finding the number of dots. Discuss with them the links between these side lengths, the basic multiplication fact that can be employed, and the units involved.
For example, the population of Mangaiti is represented by a 200 by 60 rectangle. The basic fact is 2 x 6 = 12, and the units are tens of hundreds, or thousands. So the answer to 200 x 60 is 12 000.

Questions **2b**, **2c**, and **2d** involve compound shapes for which students will need to add or subtract the number of dots in different rectangles. For example, to find the population of Waikakariki, they might find the number of dots in a complete 80 by 40 rectangle (3 200) and subtract the dots that would be in the missing rectangle (20 x 20 = 400).

Questions **3** and **4** are about the effect of changing the scale factor and the use of scale factors that are appropriate to a situation. The effect of making each dot equal to 10 is equivalent to the operation "multiply by 10".

For example:

| | |
|---|---|
| 1 dot represents 1 person (40 by 50 array) | 2 000 people |
| 1 dot represents 10 people (40 by 50 array) | 20 000 people |
| 1 dot represents 100 people (40 by 50 array) | 200 000 people |

In answering question **4**, students need to consider the implications of their choice of scale. If the numbers are very large, a suitable scale must be selected if the display is to be practical. For example, the population of the Earth is about 6.5 billion people.
If each dot represents 10 people, 600 000 000 dots are needed; that's an array of 30 000 x 20 000 dots, which is obviously not practical.
If each dot represents 1 000 people, 6 000 000 dots are needed; that's an array of 3 000 x 2 000 dots.
If each dot represents 1 000 000 people, 6 000 dots are needed; that's an array of 300 x 20 or 150 x 40 or 75 x 80. That display works!

**Links**

**Numeracy Project materials**

(see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 6: Teaching Multiplication and Division*
  Dotty Arrays, page 37

**Figure It Out**

- *Number: Book Three,* Levels 3–4
  Number Patterns, pages 8–10
  How Many?, pages 12–13
- *Number: Book Five,* Years 7–8, Level 4
  Plastic Fantastic, page 17
- *Number Sense: Book One,* Years 7–8, Link
  Keep Your Shirt On, page 23
- *Number Sense: Book Two,* Years 7–8, Level 4
  No Space to Spare, page 18

## Pages 22–24: Rolling Up!

**Achievement Objectives**

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

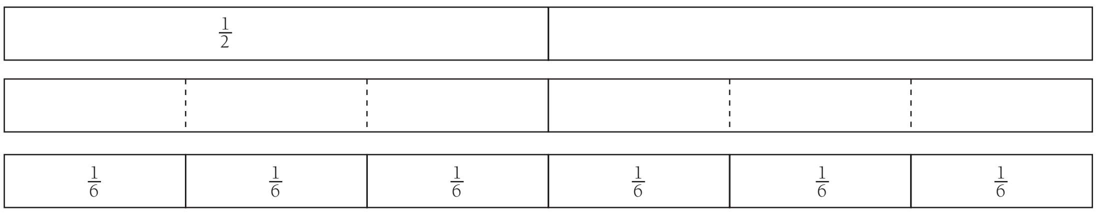
**Number Framework Links**

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

### Activity

The problems in this activity look at ways to tell if one fraction is larger or smaller than another fraction. Many misconceptions about fractions come from generalisations about whole numbers. For example, a student might say that $\frac{3}{4}$ is larger than $\frac{2}{3}$ because 3 is larger than 2. This is a correct answer for the wrong reason.

Questions **1** and **2** are aimed at establishing students' knowledge about the proximity of fractions to $\frac{1}{2}$. One-half is a key "benchmark" fraction for students. Begin by considering what fractions equal $\frac{1}{2}$. "Splitting" is a useful approach. For example, if students split a strip into 2 equal parts and then split one of those parts into 3, they show that $\frac{1}{2} = \frac{3}{6}$:




Students need to be able to make the generalisation that for all fractions equal to $\frac{1}{2}$, twice the numerator (top number) equals the denominator (bottom number) and half the denominator equals the numerator. For example:

$\times 2 \; \frac{5}{10} \; \div 2$

This knowledge allows the students to compare the strategies used by Rolf, Moana, and Jake to tell if a fraction is greater or less than $\frac{1}{2}$. All three strategies work, but Rolf's and Jake's methods are more efficient for fractions that have large numbers for the numerator and denominator. For example, to check $\frac{47}{92}$, Moana must make up a list of nearly 50 fractions. Rolf doubles 47 to get 94. 94 is greater than 92, so the fraction is larger than $\frac{1}{2}$ as $\frac{47}{94}$ equals $\frac{1}{2}$.

Algebraically, this means that the fraction $\frac{a}{b}$ is greater than $\frac{1}{2}$ if $2a > b$ or $\frac{b}{2} < a$.

In questions **3** and **4**, the students need to generalise these ideas so that they can be applied to the fractions $\frac{3}{4}$ and $\frac{2}{3}$. Start by generating some fractions equivalent to $\frac{3}{4}$ and $\frac{2}{3}$ so that the students can look for common number properties.

Using splitting, $\frac{1}{3}$ split into equal pieces gives $\frac{2}{6}$, so $\frac{2}{3}$ is equal to $\frac{4}{6}$:

| | | |
|---|---|---|
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

So the fractions $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, … are all equivalent to $\frac{2}{3}$. The number properties common to them can be expressed algebraically in these ways:

- Half the numerator multiplied by 3 equals the denominator; that is, given $\frac{a}{b} = \frac{2}{3}$, then $3(\frac{a}{2}) = b$ or $\frac{3a}{2} = b$.
- One-third of the denominator multiplied by 2 equals the numerator; that is, if $\frac{a}{b} = \frac{2}{3}$, then $\frac{2b}{3} = a$.

To test if a fraction is equal to $\frac{2}{3}$, Piripi and Marissa create an equivalent fraction against which to compare it. Piripi operates on the denominator and Marissa on the numerator.

To test $\frac{7}{12}$, Piripi makes a fraction equivalent to $\frac{2}{3}$ with 12 as the denominator; that is, $\frac{2}{3} = \frac{8}{12}$. $\frac{7}{12} < \frac{8}{12}$, so it is smaller than $\frac{2}{3}$.

Marissa makes a fraction equivalent to $\frac{2}{3}$ with 7 as the numerator; that is, $7 \div 2 = 3\frac{1}{2}$ and $3\frac{1}{2} \times 3 = 10\frac{1}{2}$, so $\frac{7}{10\frac{1}{2}} = \frac{2}{3}$ and $\frac{7}{12} < \frac{7}{10\frac{1}{2}}$, so $\frac{7}{12}$ is smaller than $\frac{2}{3}$.

Question **6** is about whether the order of multiplication and division operations makes any difference to the result. It doesn't. For example, $12 \times 3 \div 4$ and $12 \div 4 \times 3$ give the same result (9). Note that 9 is $\frac{3}{4}$ of 12.

So, to find a fraction ($\frac{a}{b}$) of a number $n$, you can either work out $n \div b \times a = \square$ or $n \times a \div b = \square$. In any given situation, it is likely that one of these orders will be more convenient than the other, so students need to understand that they can choose freely.

The **Challenge** section is about measuring one fraction by another, which is equivalent to division by a fraction. This is a very difficult concept even for students who are at the advanced proportional stage.

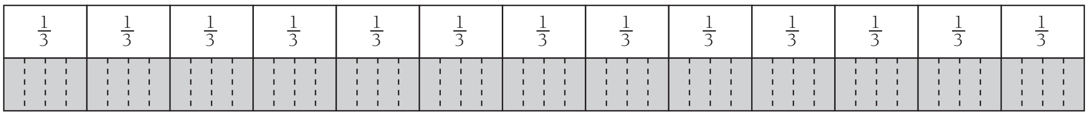
For example, to find how many $\frac{2}{3}$ rolls are in $\frac{13}{3}$:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| | | | | | | | | | | | | |

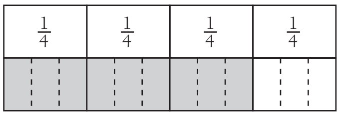
2 goes into 13 six times, so with 1 remainder, you can get $6\frac{1}{2}$ two-third rolls in $\frac{13}{3}$. This is equivalent to $\frac{13}{3} \div \frac{2}{3} = 6\frac{1}{2}$.

To find how many $\frac{3}{4}$ lengths fit into $\frac{13}{3}$ is more difficult because the fractions have different denominators (the units are different). One strategy is to rename each length in the same unit:

Splitting each third into 4 equal parts gives $\frac{52}{12}$.




Splitting each quarter into 3 equal pieces gives $\frac{9}{12}$ for $\frac{3}{4}$.




It is now possible to measure $\frac{52}{12}$ with $\frac{9}{12}$. $52 \div 9 = 5\frac{7}{9}$, which is how many $\frac{3}{4}$ length rolls can be made from a $\frac{13}{3}$ roll. This could be written as: $\frac{13}{3} \div \frac{3}{4} = \frac{52}{12} \div \frac{9}{12} = 5\frac{7}{9}$.

**Links**

**Numeracy Project materials**

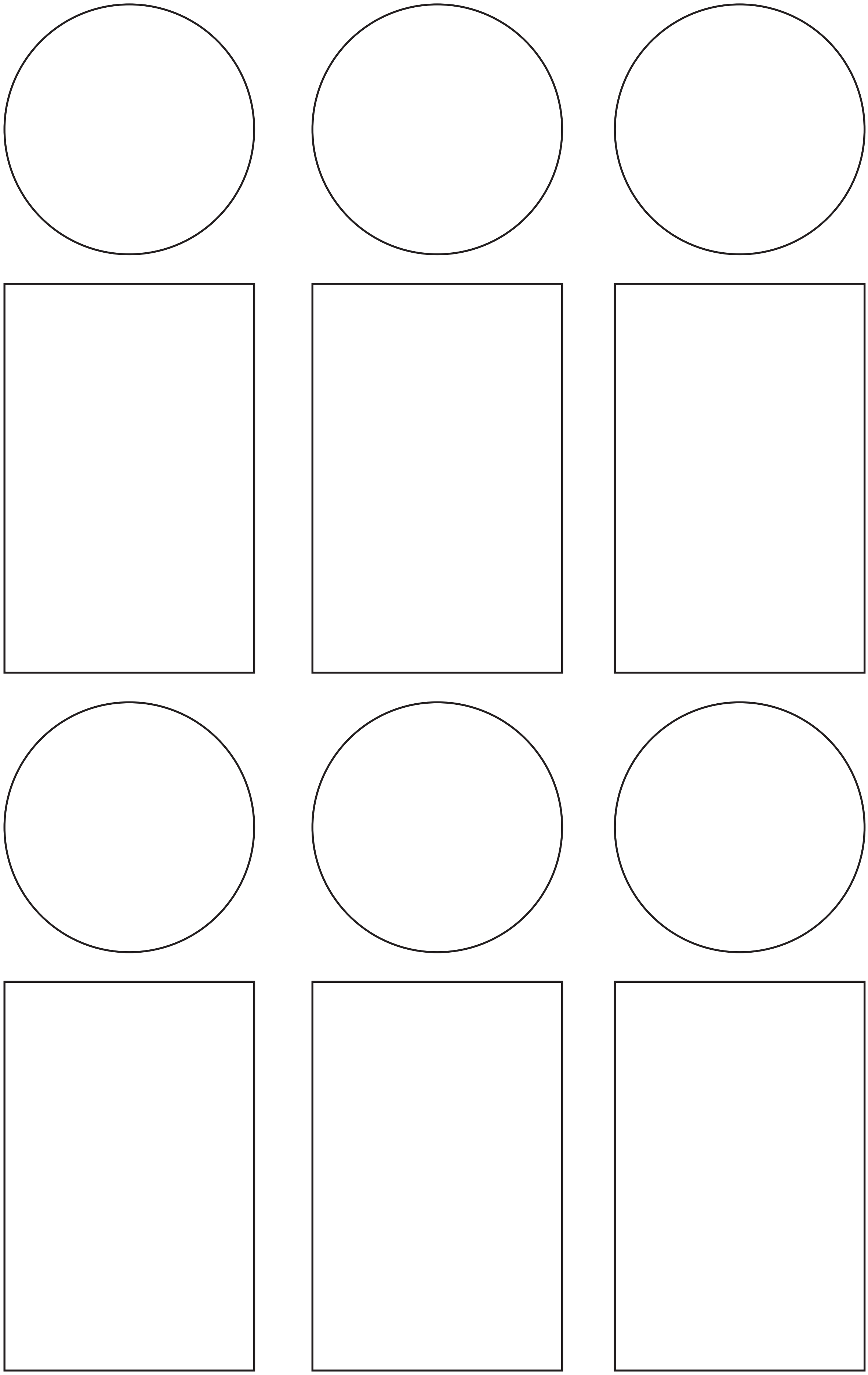
(see www.nzmaths.co.nz/numeracy/project_material.aspx)

- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Estimating with Fractions, page 15
  Equivalent Fractions, page 16
  Fractions, page 16
  Dividing Fractions, page 21

**Figure It Out**

- *Number Sense and Algebraic Thinking: Book One,* Levels 3–4
  Close Ties, pages 14–15

## Copymaster: Cutting It



# Copymaster: The Power of 10

## Copymaster: 1-200 Chart

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 |
| 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 |
| 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160 |
| 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 |
| 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 |
| 191 | 192 | 193 | 194 | 195 | 196 | 197 | 198 | 199 | 200 |

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