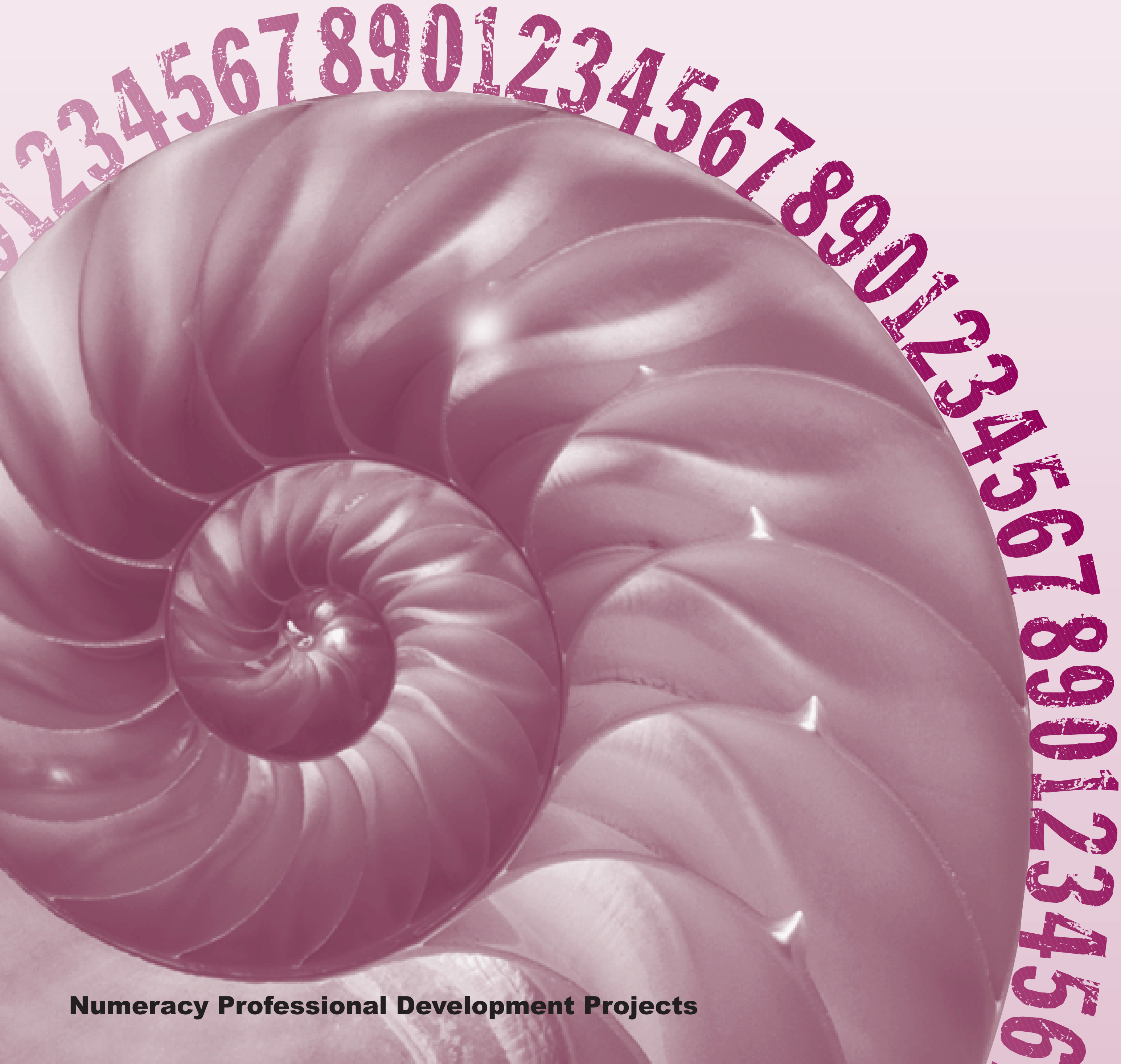
MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Book 6

Teaching Multiplication and Division

Revised Edition 2007



Numeracy Professional Development Projects

EFFECTIVE MATHEMATICS TEACHING

The Numeracy Professional Development Projects assist teachers to develop the characteristics of effective teachers of numeracy.

Effective teachers of numeracy demonstrate some distinctive characteristics.[1] They:

- have high expectations of students' success in numeracy;
- emphasise the connections between different mathematical ideas;
- promote the selection and use of strategies that are efficient and effective and emphasise the development of mental skills;
- challenge students to think by explaining, listening, and problem solving;
- encourage purposeful discussion, in whole classes, in small groups, and with individual students;
- use systematic assessment and recording methods to monitor student progress and to record their strategies for calculation to inform planning and teaching.

The focus of the Numeracy Professional Development Projects is number and algebra. A key component of the projects is the Number Framework. This Framework provides us as teachers with:

- the means to effectively assess students' current levels of thinking in number;
- guidance for instruction;
- the opportunity to broaden our knowledge of how children acquire number concepts and to increase our understanding of how we can help children to progress.[2]

The components of the professional development programme allow us to gather and analyse information about children's learning in mathematics more rigorously and respond to children's learning needs more effectively. While, in the early stages, our efforts may focus on becoming familiar with the individual components of the programme, such as the progressions of the Framework or carrying out the diagnostic interview, we should not lose sight of the fact that they are merely tools in improving our professional practice. Ultimately, the success of the programme lies in the extent to which we are able to synthesise and integrate its various components into the art of effective mathematics teaching as we respond to the individual learning needs of the children in our classrooms.

2007 Revised Edition
Book 6 includes the following modifications and additions:

- Key mathematical ideas are now directly linked to specific lessons rather than to an overall stage. This is to support teachers with the mathematical content of the lesson.
- Diagnostic Snapshots are provided for use at the start of a lesson to determine the readiness of students for the ideas in that lesson. These are indicated by the 📷 icon.
- The key mathematics that needs to be developed simultaneously with students while they are working through a strategy stage has been identified. This is to avoid gaps in knowledge and understanding that may prevent the students from progressing to the next stage.
- Guidance in the development of written recording has been provided through indicating when teacher modelling is appropriate and suggesting forms that the written recording may take.
- There are three new lessons, which focus on multiplicative aspects of place value.

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[1] Askew et al (2000) *Effective Teachers of Numeracy*. London: King's College.

[2] See also the research evidence associated with formative assessment in mathematics: Wiliam, Dylan (1999) "Formative Assessment in Mathematics" in *Equals*, 5(2); 5(3); 6(1).

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Note: Teachers may copy these notes for educational purposes.

This book is also available on the New Zealand Maths website, at www.nzmaths.co.nz/Numeracy/2008numPDFs/pdfs.aspx

Teaching Multiplication and Division

Teaching for Number Strategies

The activities in this book are specifically designed to develop students' mental strategies. They are targeted to meet the learning needs of students at particular strategy stages. All the activities provide examples of how to use the teaching model from *Book 3: Getting Started*. The model develops students' strategies between and through the phases of *Using Materials*, *Using Imaging*, and *Using Number Properties*.

Each activity is based on a specific learning outcome. The outcome is described in the "I am learning to ..." statement in the box at the beginning of the activity. These learning outcomes link to the planning forms online at www.nzmaths.co.nz/numeracy/Planlinks/

The following key is used in each of the teaching numeracy books. Shading indicates which stage or stages the given activity is most appropriate for. Note that CA, Counting All, refers to all three *Counting from One* stages.

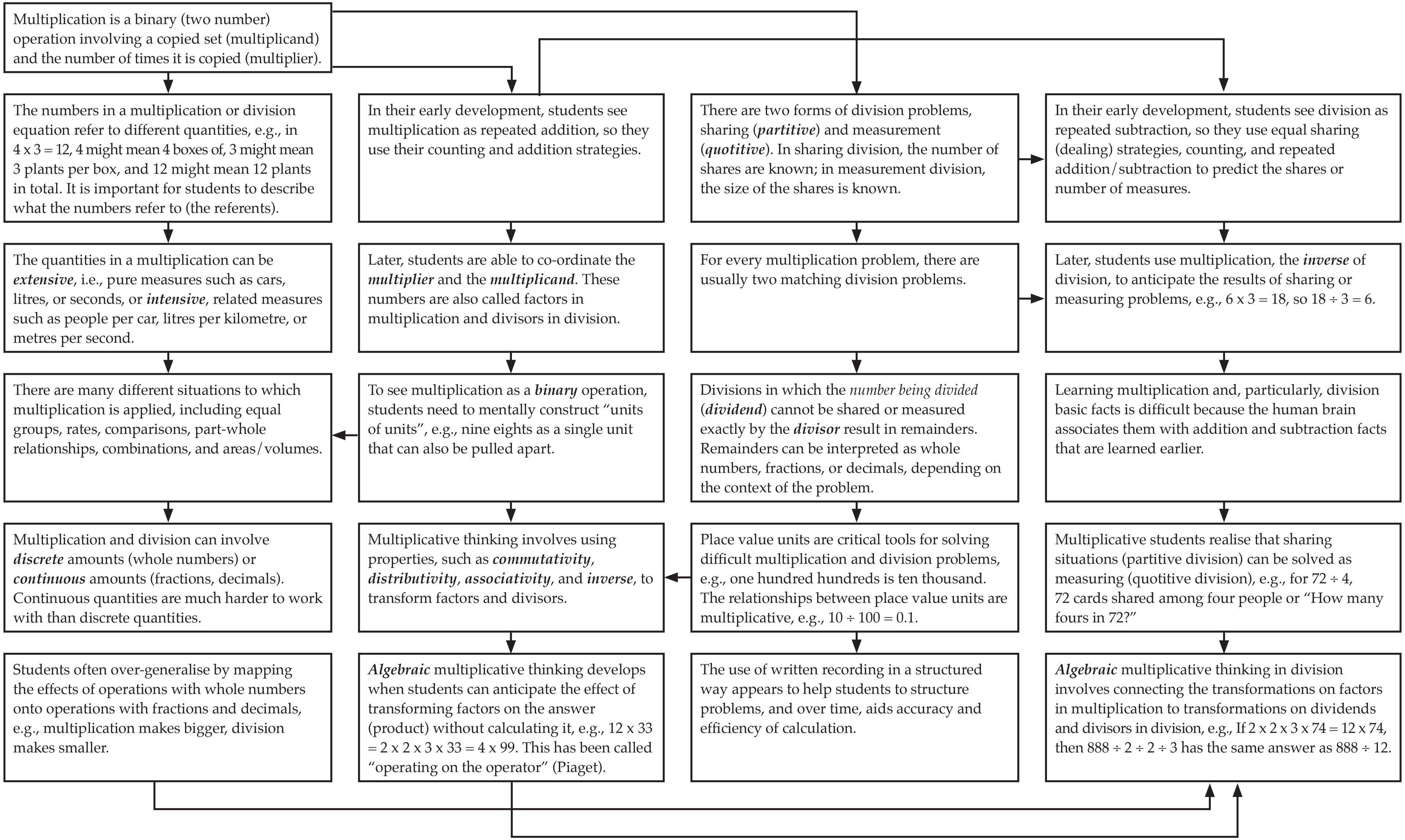
Strategy Stage

E →	Emergent
CA →	Counting All (One-to-one Counting, Counting from One on Materials or by Imaging)
AC →	Advanced Counting
EA →	Early Additive Part-Whole
AA →	Advanced Additive–Early Multiplicative Part-Whole
AM →	Advanced Multiplicative–Early Proportional Part-Whole
AP →	Advanced Proportional Part-Whole

The table of contents below details the main sections in this book. These sections reflect the strategy stages as described on pages 15–17 of *Book One: The Number Framework*. The Knowledge and Key Ideas sections provide important background information for teachers in regard to the development of students' thinking in multiplication and division.

Strategy Stage/s	Knowledge and Key Ideas	Multiplication	Division
Counting from One by Imaging to Advanced Counting	Pages 7–8	Pages 8–10	
Advanced Counting to Early Additive	Pages 11–12	Pages 12–16, 21–23	Pages 17–20
Early Additive to Advanced Additive–Early Multiplicative	Page 24–25	Pages 25–38	Pages 38–40
Advanced Additive–Early Multiplicative to Advanced Multiplicative–Early Proportional	Pages 41–43	Pages 43–48, 49–54, 67–70, 73–75	Pages 43–48, 54–67, 70–72
Advanced Multiplicative–Early Proportional to Advanced Proportional	Page 76	Pages 76–79	Pages 76–79

Key ideas in Multiplication and Division



Multiplication is a binary (two number) operation involving a copied set (multiplicand) and the number of times it is copied (multiplier).

The numbers in a multiplication or division equation refer to different quantities, e.g., in 4 x 3 = 12, 4 might mean 4 boxes of, 3 might mean 3 plants per box, and 12 might mean 12 plants in total. It is important for students to describe what the numbers refer to (the referents).

The quantities in a multiplication can be ***extensive***, i.e., pure measures such as cars, litres, or seconds, or ***intensive***, related measures such as people per car, litres per kilometre, or metres per second.

There are many different situations to which multiplication is applied, including equal groups, rates, comparisons, part-whole relationships, combinations, and areas/volumes.

Multiplication and division can involve ***discrete*** amounts (whole numbers) or ***continuous*** amounts (fractions, decimals). Continuous quantities are much harder to work with than discrete quantities.

Students often over-generalise by mapping the effects of operations with whole numbers onto operations with fractions and decimals, e.g., multiplication makes bigger, division makes smaller.

In their early development, students see multiplication as repeated addition, so they use their counting and addition strategies.

Later, students are able to co-ordinate the ***multiplier*** and the ***multiplicand***. These numbers are also called factors in multiplication and divisors in division.

To see multiplication as a ***binary*** operation, students need to mentally construct "units of units", e.g., nine eights as a single unit that can also be pulled apart.

Multiplicative thinking involves using properties, such as ***commutativity***, ***distributivity***, ***associativity***, and ***inverse***, to transform factors and divisors.

Algebraic multiplicative thinking develops when students can anticipate the effect of transforming factors on the answer (product) without calculating it, e.g., 12 x 33 = 2 x 2 x 3 x 33 = 4 x 99. This has been called "operating on the operator" (Piaget).

There are two forms of division problems, sharing (***partitive***) and measurement (***quotitive***). In sharing division, the number of shares are known; in measurement division, the size of the shares is known.

For every multiplication problem, there are usually two matching division problems.

Divisions in which the *number being divided* (***dividend***) cannot be shared or measured exactly by the ***divisor*** result in remainders. Remainders can be interpreted as whole numbers, fractions, or decimals, depending on the context of the problem.

Place value units are critical tools for solving difficult multiplication and division problems, e.g., one hundred hundreds is ten thousand. The relationships between place value units are multiplicative, e.g., 10 ÷ 100 = 0.1.

The use of written recording in a structured way appears to help students to structure problems, and over time, aids accuracy and efficiency of calculation.

In their early development, students see division as repeated subtraction, so they use equal sharing (dealing) strategies, counting, and repeated addition/subtraction to predict the shares or number of measures.

Later, students use multiplication, the ***inverse*** of division, to anticipate the results of sharing or measuring problems, e.g., 6 x 3 = 18, so 18 ÷ 3 = 6.

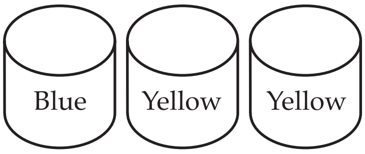
Learning multiplication and, particularly, division basic facts is difficult because the human brain associates them with addition and subtraction facts that are learned earlier.

Multiplicative students realise that sharing situations (partitive division) can be solved as measuring (quotitive division), e.g., for 72 ÷ 4, 72 cards shared among four people or "How many fours in 72?"

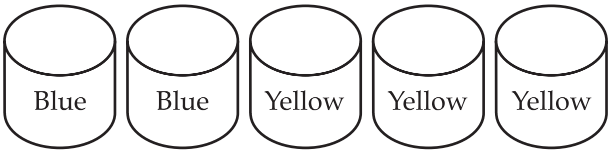
Algebraic multiplicative thinking in division involves connecting the transformations on factors in multiplication to transformations on dividends and divisors in division, e.g., If 2 x 2 x 3 x 74 = 12 x 74, then 888 ÷ 2 ÷ 2 ÷ 3 has the same answer as 888 ÷ 12.

Multiplicative Thinking

The ideas in this book are designed to help students to become multiplicative thinkers. It is well known that students apply additive thinking to situations in which multiplication is the best approach. For example, additive thinkers sometimes believe that these paint mixes produce the same colour because the difference between yellow and blue quantities is the same (i.e., one more yellow than blue):



Mixture A (1:2)

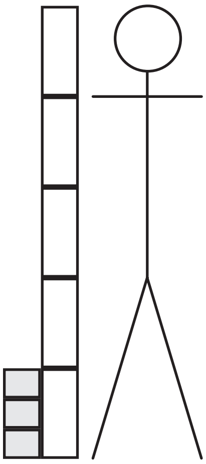


Mixture B (2:3)

Multiplication and division are critical foundations for more difficult concepts in number, algebra, measurement, and statistics. Thinking multiplicatively involves many different mathematical ideas as well as constructing and manipulating factors (the numbers that are multiplied) in response to a variety of contexts. For example, suppose a student is asked to measure the height of a stick figure in light and grey rods.

A student thinking multiplicatively uses the longer light rod as their initial measure. Realising that three grey rods equals one light rod, the student constructs the factors 5 × 3 as a way to model the problem of how many grey rods measure the figure.

Being multiplicative also involves having key items of knowledge, for example, basic facts such as 5 × 3 = 15, available as points of leverage to find unknown results. Deriving from known facts using the properties of multiplication and division is a key feature of multiplicative thinking.



Vocabulary

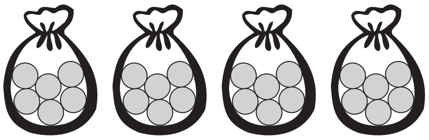
While it is not necessary for students to know the mathematical terms for these properties, it is important that they have a vocabulary to describe the strategies they use. Below is a list of common properties for multiplication and division and some suggested vocabulary. Students firstly need to know that the numbers being multiplied are called factors, e.g., 4 × 8 = 32, so 4 and 8 are factors of 32, so 32 will divide exactly by 4 and 8.

Property	**Example**	**Suggested Vocabulary**
Commutative	8 × 5 = 5 × 8	Changing the order of factors
Distributive	4 × 23 = 4 × 20 + 4 × 3 7 × 28 = 7 × 30 – 7 × 2	Splitting factors Tidy numbers
Associative	12 × 25 = 3 × (4 × 25) = 3 × 100	Doubling and halving Thirding and trebling, etc.
Inverse	42 ÷ 3 = as □ × 3 = 42 14 × 3 = 42 so 42 ÷ 3 = 14	Reversing Doing and undoing

Contexts for Multiplication and Division

There are many different contexts to which multiplicative thinking can be applied. While mathematically they all involve the operations of multiplication and division, to students they appear to be quite different. In each type of problem, the factors and answers (products) refer to different kinds of quantities. For this reason, it is important for students to say and record these "referents" when solving problems in context.

Consider the most commonly used type of multiplication and division problems, those involving equal sets. You might ask the students to solve the problem, "Kayla has four bags of marbles. There are six marbles in each bag. How many marbles does Kayla have?"



Traditionally, this problem is modelled by the equation, $4 \times 6 = 24$. However, attaching the referents connects the symbols to their meaning within the context: "4 bags with 6 marbles in each bag (marbles per bag) is 24 marbles in total."

Over time, students learn to form an equation from the context, carry out the operation mathematically, then map the answer back into the context. However, even at that level of sophistication, it is important that students explain what referent their answer refers to.

For every multiplication there are two types of division, partitive (sharing) and quotitive (grouping/measuring). Consider the context of 24 marbles and 4 bags. The associated division problems are:

1. "Kayla has 24 marbles. She shares them equally into 4 bags. How many marbles are in each bag?" (Sharing division)

   24 marbles shared equally into 4 bags gives 6 marbles per bag.

2. "Kayla has 24 marbles. She puts them into bags of 6 marbles. How many bags can she make?" (Measurement division; 24 is grouped/measured in sixes.)

   24 marbles put into bags of 6 marbles per bag gives 4 bags.

Common Types of Contexts

The table on page 5 presents the common types of contexts for multiplication and division. Each type is presented as multiplication and the two forms of division. A powerpoint workshop on problem types is available at nzmaths: www.nzmaths.co.nz/numeracy/Other%20material/Tutorials/Tutorials.aspx

Problem Types

Problem Type	Multiplication	Division by Sharing	Division by Measuring/ Grouping
Equal groups	5 taxis have 4 passengers in each. How many passengers are there altogether? $5 \times 4 = 20$	20 passengers have to fit equally into 5 taxis. How many passengers go in each taxi? $20 \div 5 = 4$	20 passengers travel 4 to a taxi. How many taxis are needed? $20 \div 4 = 5$
Rate	You can wash 1 window in 4 minutes. How long will it take to wash 5 windows? $5 \times 4 = 20$	You can wash 5 windows in 20 minutes. How long does it take to wash 1 window? $20 \div 5 = 4$	How many windows can you wash in 20 minutes if you wash 1 window every 4 minutes? $20 \div 4 = 5$
Comparison	Henry has six times as many marbles as Sarah. Sarah has 12 marbles; how many marbles does Henry have? $6 \times 12 = 72$	Henry has six times as many marbles as Sarah. Henry has 72 marbles; how many does Sarah have? $72 \div 6 = 12$	Henry has 72 marbles and Sarah has 12 marbles. How many times more marbles does Henry have than Sarah? $72 \div 12 = 6$
Part-Whole	For every 3 boys in the class, there are 4 girls. There are 35 students in the class. How many are girls? $3 + 4 = 7$ (3:4) $5 \times 7 = 35$ $5 \times 4 = 20$	For every 3 boys in the class, there are 4 girls. There are 20 girls in the class. How many boys are in the class? $20 \div 4 = 5$ $5 \times 3 = 15$	In a class of 35 students, 15 are boys. What is the ratio of boys to girls? $35 - 15 = 20$ 15:20 is the same as 3:4
Cartesian Product	Simon has 4 T-shirts and 3 pairs of shorts. How many *different* T-shirt and short outfits can he make? $4 \times 3 = 12$	Simon can make 12 different outfits of a T-shirt with a pair of shorts. He has 4 T-shirts. How many pairs of shorts does he have? $12 \div 4 = 3$ Or: He has 3 pairs of shorts. How many shirts does he have? $12 \div 3 = 4$	
Rectangular Area	A rectangle has sides that are 8 and 7 units long. What is the area of the rectangle? $7 \times 8 = 56$ sq. units	A rectangle has an area of 56 square units. One side is 8 units long. What is the length of the other side? $56 \div 8 = 7$ Or: One side is 7 units long. What is the length of the other side? $56 \div 7 = 8$	

A powerpoint workshop on Multiplicative Thinking: situations for multiplication and division, is available through this link: www.nzmaths.co.nz/Numeracy/Other%20material/Tutorials/Tutorials.aspx

The Number Framework is a synthesis of the research into students' learning progressions in number. Links are made between how students think across the strategy domains of addition and subtraction, multiplication and division, and ratios and proportions. There is strong evidence to suggest that students' early views of multiplication are of repeated addition. It is not clear if this is due to the way we present multiplication or if repeated addition is a natural (intuitive) view developed through real life experiences. Initially, students seem to think of partitive division as equal sharing and quotitive (measurement/grouping) division as repeated subtraction. There is considerable commonality in the progressions suggested by researchers that are summarised in the table on below.

Key Progressions

Number Framework Stage/s	Descriptor	Multiplication	Division
1–1 Counting to CA imaging	Counting using units of one	Forming sets and counting the objects in ones directly or by imaging	Sharing the objects one by one or forming equal sets by ones
AC to EA	Composite counting	Skip counting or repeated addition	Anticipating sharing by repeated addition or subtraction or by skip counting
AA	Known facts and deriving	Knowing facts and using commutative, distributive, and associative properties to find unknown facts from known facts	Anticipating sharing and repeated subtraction by reversing multiplication facts
AM		Applying the properties of multiplication to a full range of contexts and whole numbers	Connecting the properties of multiplication to both forms of division using inverse operations across a full range of contexts with whole numbers
AP	Operating on the operator (thinking about the factors without needing to specify the product)	Applying the properties of multiplication to a full range of contexts with fractions and decimals and manipulating multiplicative relationships algebraically	Connecting the properties of multiplication to both forms of division using inverse operations across a full range of contexts with fractions and decimals and manipulating division relationships algebraically

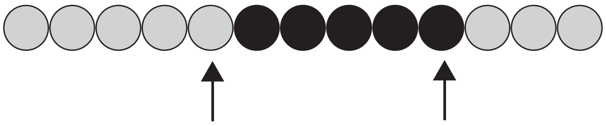
A powerpoint workshop on progressions in multiplicative thinking is available through this link:
www.nzmaths.co.nz/Numeracy/Other%20material/Tutorials/Tutorials.aspx

Teaching Multiplication and Division

Learning Experiences to Move from Counting from One by Imaging to Advanced Counting

Required Knowledge

Before attempting to develop their ideas about multiplication and division, check that Counting All students have the following knowledge.

Key Knowledge	Questions for Key Knowledge
• Forwards and backwards skip-counting sequences in twos and fives at least	Use a bead string with colour breaks of the skip counting required. Start by counting each bead and then ask the students to name each bead at the end of a colour break. Use breaks of twos, threes, fives, and tens. Plastic tags can be attached to the bead strings at the decade breaks, 0, 10, 20, ... etc.  Record the end numbers in sequence as a reference for the students: 5, 10, 15, 20, 25, ... Gradually erase numbers until the students can say the sequence, 5, 10, 15, ..., without seeing the numbers written down. For independent work, write multiples on supermarket bag tags and get the students to attach the tags to the string after the given beads.

Knowledge to be Developed

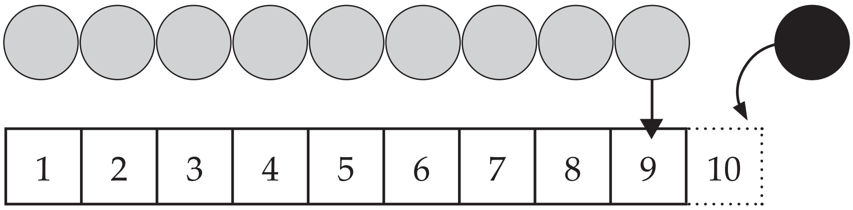
While students at the Counting All stage are developing their ideas about multiplication and division, it is essential that they have the following knowledge as a focus:

- Doubles to 20: Double Trouble, page 32, *Book 4: Teaching Number Knowledge*
- Skip counting twos, fives, and tens. See page 1 of Planning Sheet CA–AC (Mult and Div) www.nzmaths.co.nz/numeracy/Planlinks/MultPlannerCA-AC.pdf

E
CA
AC
EA
AA
AM
AP

Key Ideas

The students must realise that skip-counting sequences relate to putting sets of the same number together. This is tied to the cardinality principle that the end-point count measures the set and that rote-counting on or back gives the result of adding on more objects or taking objects off (ordinality principle).



"Nine and one more is 10."

Learning Experiences

Number Strips

I am learning to solve equal group problems by skip counting.

Key Mathematical Ideas

- Students must realise that skip-counting sequences, e.g., 3, 6, 9, relate to putting groups of the same number together.
- As students skip count, they need the understanding that any number in the counting sequence will give them the total so far (cardinality principle).
- Each successive counting number gives the result of the addition of a group of objects, e.g., with 3, 6, 9, when the student says 12, they know that they have added 3 more on and similarly for skip counting backwards, they know that they are taking 3 objects off.

Key Mathematical Knowledge

Check that the students can skip count forwards and backwards in twos and fives.

Diagnostic Snapshot

Put down counters of different colours to show, for example, 4 lots of 3 in one row. Ask the students questions such as: "How many counters are there altogether?" "If we counted them in twos, how many counters would there be?"

Students who skip count for the first question and then reply 12 to the second question without re-counting need to be extended.

Equipment: Number strip (1–100) (see Material Master 6–1), transparent coloured counters

Using Materials

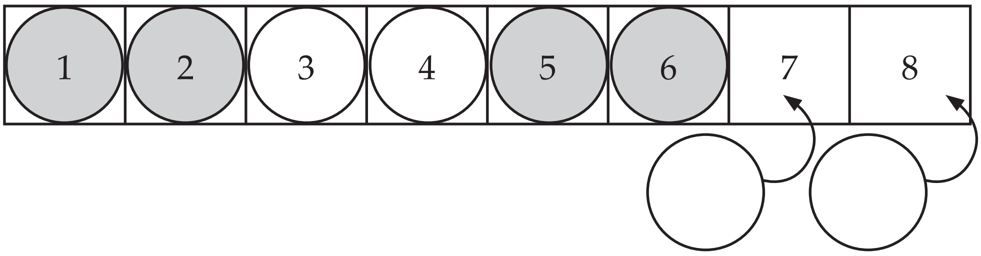
*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Lay down six sets of two counters. Make each set of counters the same colour, e.g., two reds, two yellows, two blues, etc.

Problem: "I am going to put these counters onto a number strip like this." (Put the first two counters onto the strip covering 1 and 2.) "When I put all of the counters on, what will be the last number I cover?"

Let the students try to image the process of counters being moved.

Have the students take turns placing a pair of counters onto the next numbers. Each time, ask for predictions about the end-point number:



"If I put two more counters onto the strip, how many will I have?"

When all the counters have been placed, point to the counter at the end of each colour break and get the students to say the numbers, "2, 4, 6, 8, ...". To encourage imaging, turn the strip over so that the numbers are not visible and replace the counters. Point to given counters and ask the students to tell you what number would be under the counter.

Record the operation using the symbols 6 × 2 = 12.

Ask predictive questions such as "I have six sets of two. How many counters would I have if I put on/took off a set of two?" Record the result using symbols, 7 × 2 = 14 or 5 × 2 = 10.

Pose similar problems with counters and the number strip, such as:
"Five sets of three" (5 × 3)   "Four sets of four" (4 × 4)

Using Imaging

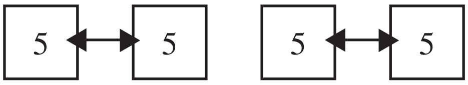
Shielding: Problem: Turn over the number strip so that the numbers are face down. Ask four students to get five counters of one colour and hide them under their hands. Confirm the fiveness of each set by getting the students to lift their hands and then replace them.

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

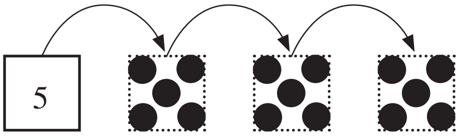
Ask "How many counters have we got altogether under these hands?" "If I put all of the counters on the number strip, what will be the last number I cover?"

Let the students try to solve the problem through imaging. Focus them on using doubles or skip-counting knowledge:

Example 1: "How many counters are in these two hands altogether?" (five and five, that's 10) "How many will be in the next two hands?" (another 10) "How many is that altogether?" (10 and 10 is 20)



Example 2: "How many counters are under this hand?" (five) "Imagine I put them on the number strip. Then add another five." (lifting the hand and then replacing it) "How many is that so far on the number strip?" "And another five?" "And another five?"



If necessary, fold back into materials by putting the sets of counters onto the number strip. Record the results using equations, for example, 4 × 5 = 20 or 5 + 5 + 5 + 5 = 20. Ask the students how many counters there would be if another five were added or taken away. Encourage them to build on from the previous total rather than counting from one.

Pose similar problems using imaging, ensuring the numbers chosen are accessible for skip counting.

Nine sets of two (9 × 2 = 18) then 8 × 2 = 16 and 10 × 2 = 20

Three sets of three (3 × 3 = 9) then 4 × 3 = 12 and 5 × 3 = 15

Using Number Properties

The students have a good understanding of Counting On when they can use skip-counting sequences to solve multiplication problems. Write multiplication equations and discuss what they mean.

For example, 6 × 2 = ☐ means, "six sets of two is the same as what?"

Other suitable examples are: 7 × 5 = ☐ 8 × 2 = ☐ 5 × 10 = ☐ 6 × 3 = ☐

Independent Activity

Play a game of Hit the Spot. The students need counters, a number strip (Material Master 6–1) and a 1–6 dice. Decide on the spot number. 10, 12, 15, 16, 18, 20, and 24 are good choices as they have many factors. The players take turns to roll the dice (say, three comes up). If they can make the spot number exactly, in sets of that dice number (three), they get a point. So, in this case, four sets of three equals 12. The students can model the sets with counters if they need to (putting four sets of three on the strip will reach 12 exactly). The winner is the student with the most points after five dice rolls each.

Learning Experiences to Move Students from Advanced Counting to Early Additive Part-Whole

Required Knowledge

Before attempting to develop their ideas about multiplication and division, check that Advanced Counting students have the following knowledge.

Key Knowledge	Questions for Key Knowledge
• Forwards and backwards skip-counting sequences in twos, fives, and tens at least	Use the addition and subtraction constant function on a calculator to generate skip-counting sequences. Keying in + 5 = = = = = produces the counting sequence in fives. Mark the multiples using transparent counters on a student hundreds board. Encourage the students to predict how many times = must be pressed to get to target numbers, like 35. Turn over multiples on the hundreds boards and look for symmetrical patterns in the multiples. For example, multiples of five form a vertical pattern, and multiples of three form a diagonal pattern. Ask the students to explain why they think the patterns occur, e.g., five divides evenly into 10.

Knowledge to be Developed

While students at the Advanced Counting stage are developing their ideas about multiplication and division, it is essential that they have the following knowledge as a focus:

- Skip counting in threes – see page 1 of Planning Sheet AC to EA (Mult and Div) www.nzmaths.co.nz/numeracy/Planlinks/MultPlannerAC-EA.pdf
- Learning multiplication and division facts for 2, 5, and 10 times tables – see page 2 of Planning Sheet EA to AA (www.nzmaths.co.nz/numeracy/Planlinks/MultPlannerEA-AA.pdf) and page 2 of AC to EA (Mult and Div)

Key Ideas

Students at the Advanced Counting stage are learning to use addition strategies to solve problems that could be solved by multiplication and division. For example, given four sets of five, to find the total number, they may add five and five to get 10 and then add 10 and 10. These students are learning to form the numbers (factors) to multiply and divide by for these situations.

It's important that the students learn the connection between multiplication and repeated addition, for example, 5×4 is the same as $4 + 4 + 4 + 4 + 4$, and that changing the order of the factors gives the same result, for example, $5 \times 4 = 4 \times 5$ or $5 + 5 + 5 + 5$. Students need to know why this commutative property is true and to have generalised this property to relate to larger numbers, e.g., $3 \times 99 = 99 \times 3$.

The students also need to understand two different types of division situations. These two types are equal sharing, as in 12 lollies shared between four people, and measuring, as in 12 lollies put into sets (or bags) of three.

Linking these ideas to everyday contexts that the students are familiar with may assist their understanding. Using everyday vocabulary, such as “groups of”, “teams of”, and “packets and piles of”, alongside the mathematical terminology and “sets of” is helpful to the students.

To avoid ambiguity, clarify the language used. For example, by “five times three” we mean “five sets of three”, although its original meaning is “five replicated three times”. You need to be aware of potential confusion with interpretations of multiplication statements. In mainstream classes in New Zealand, the first number in a multiplication expression represents the multiplier and the second number represents the multiplicand, e.g., in 5 × 3, 5 is the number of equal sets and 3 is the size of each set. This is a convention that is not shared across all cultures. In Korean and in Māori immersion classes, 5 × 3 is regarded as “5, three times”.

Learning Experiences

Three’s Company

I am learning to solve “three times” problems.

Key Mathematical Ideas

- Multiplication is a binary operation, i.e., it acts on two numbers (bi- means two). In the operation 6 × 3, 6 is the multiplier (how many of) and 3 is the multiplicand (of what).
- Students at first see multiplication as repeated addition and use their addition strategies to solve multiplication problems, i.e., they don’t see it as a binary operation (an operation involving two numbers), e.g., they see 6 × 3 as 3 + 3 + 3 + 3 + 3 + 3.
- The learning of multiplication facts is a critical part of helping students progress towards multiplicative thinking. It is from known facts that the new results can be derived using multiplicative properties.

Key Mathematical Knowledge

Check that the students know the following:

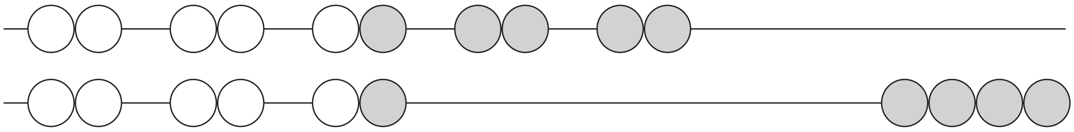
- The forward skip-counting sequences in twos, fives, and tens, e.g., 5, 10, 15, …
- That skip-counting sequences measure the number of single items in collections of equal sets, e.g., five equal sets of three.

Diagnostic Snapshot

Ask the students to count:

- forward in twos to twenty
- forward in fives to fifty
- forward in tens to one hundred.

Show the students collections of objects arranged in sets of two, five, or ten and ask them to count the objects, e.g., beads on the Slavonic abacus.



Check to see that the students use skip counting to find out how many objects are counted in total.

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Equipment: Cards or supermarket bag tags labelled with the numbers in the skip-counting sequences for twos, threes, fives, and tens, a 100-bead string or Slavonic abacus, a hundreds board, calculators

Using Materials

1. Put the cards or tags for the skip-counting sequence for twos (2, 4, 6, , …) randomly on the mat and ask the students what pattern they can see in the numbers. Expect them to put the numbers in counting sequence.

   Ask them to perform a counting sequence in ones, with the accent on the numbers in front of them, i.e., 1, **2**, 3, **4**, 5, **6**, 7, **8**, …, while you move beads one at a time across the bead string or abacus.

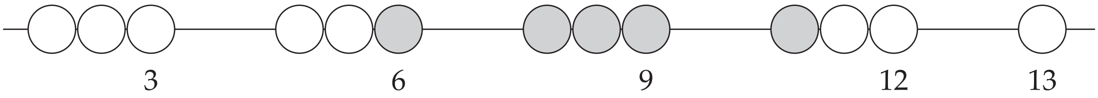
   Next, ask them to perform a mime count on the odd numbers, only saying the twos sequence numbers, i.e., _, 2, _, 4, _, 6, _, 8, …, while you move single beads across.

   Next, ask them to say only the twos skip-counting sequence as you move groups of two beads across.

   Finally, get them to find the bead that matches a given twos count number, e.g., “Put this tag after bead 12.” Give the numbers randomly so that the students can count from previous numbers, e.g., to find bead 8, a student might count in twos from bead 4.

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

2. Repeat this sequence with the skip-counting sequences for ten and five.
3. Put the tags or cards for the skip-counting sequence by threes on the mat in mixed order. Ask the students what counting pattern exists in these numbers. Work through the same sequence as above with the three sequence.
4. Move over sets of three on the bead string or abacus but add one or two extra beads on the end. Ask the students to tell you how many beads have been moved in total. Look for them to recognise that the skip sequence works for groups of three but the remainders need to be counted in another unit, one or two.



Using Imaging

Tell the students that you going to count in threes but they will only be able to hear the sets of three being moved. Hide the bead string or turn the abacus around. Make sure that each three makes an audible “clack” as it is moved. Challenge the students by moving a different number of sets of three each time. Encourage the students to say the counting sequence quietly to themselves so that they only announce the answer if asked or write it invisibly on the palm of their hand for only you to see.

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Progress to telling the students how many sets of three you have moved in one go. Bring in the vocabulary of multiplication, e.g., “So four sets of three is twelve, four times three is twelve.”

Next, ask the students to visualise the sets of three as divisors. Choose a counting sequence number, say 15, move that number of beads across and ask, “How many threes are in fifteen?”

In all steps of this process, the bead string or abacus can be exposed when necessary to check on answers and ideas.

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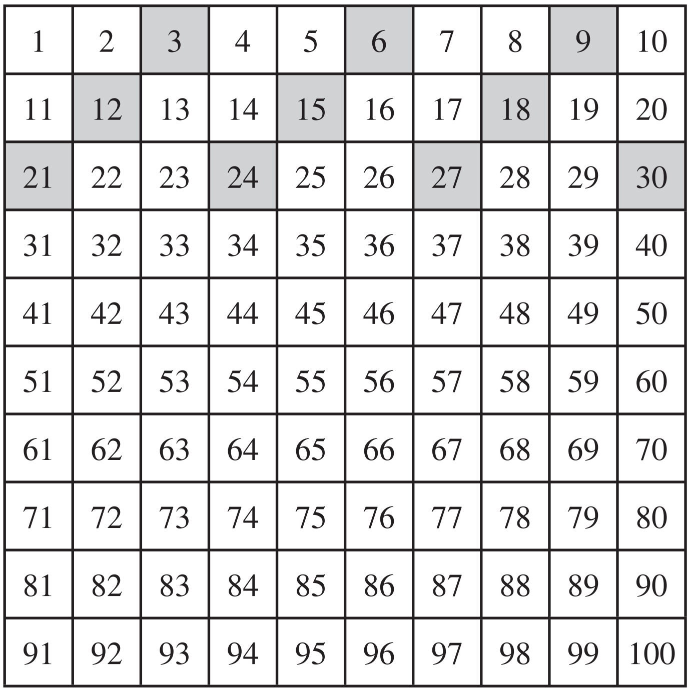
Using Number Properties

Look at the patterns that skip-counting sequence numbers make on a hundreds chart, e.g., the sequence of threes makes a diagonal pattern like this:



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Ask the students to predict other numbers in the threes sequence. If necessary, these can be checked by counting up the sequence.

Ask questions such as, "Will 41 be in the pattern? How do you know?"

Point to numbers in the sequence (30 or less) and ask, "How many threes are in that number?"

Ask the students to say or write the matching multiplication statement, e.g., "Seven times three is twenty-one" or 7 × 3 = 21.

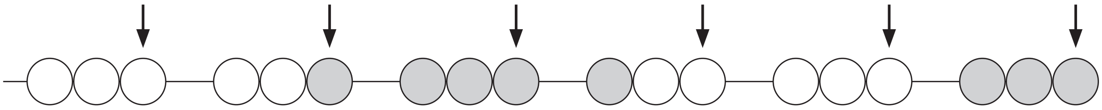
Students can practise the counting sequences in pairs with a calculator. If they enter + 3 and press the equals sign repeatedly, this will generate the sequence 3, 6, 9, 12, … as the calculator adds three repeatedly. Similarly, entering a number minus three, e.g., 30 – 3, then pressing equals generates the repeated subtraction sequence, 30, 27, 24, 21, …

Have the students set challenges for a classmate such as:

"Press + 3. Without looking, press equals until 18 is in the window."

"Press 15 – 3. Without looking, press equals until zero is in the window."

Note: Students may realise that the answers to sets of these problems can be found using sets of two, e.g.:



6 sets of 3 is 6 sets of 2 plus 6 extra ones, i.e., 6 × 3 = (6 × 2) + 6.

This shows an early understanding of the distributive property.

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Animal Arrays

I am learning to find other ways to solve problems like 5 + 5 + 5.

Key Mathematical Ideas

- Students are learning the connection between repeated addition and multiplication, e.g., 4 + 4 + 4 + 4 + 4 is the same as 5 × 4.
- Students are introduced to an array model that visually lends itself to identifying equal rows or columns. The convention in New Zealand is to regard 4 × 3 as four sets of three.
- Given an expression such as 5 × 4, the student is able to create an array and then derive other multiplication facts from this, e.g., 6 × 4 by adding on a row of 4 and 4 × 4 by subtracting a row of 4.
- The language of multiplication is introduced, e.g., "rows of", "lots of", and "groups of". The connection between the addition and multiplication equation is made, e.g. five plus five plus five is fifteen is the same as three rows of five. 5 + 5 + 5 can be recorded as 3 × 5.

Key Mathematical Knowledge

Check that the students can skip count forwards and backwards in twos, fives, and tens.

Diagnostic Snapshot

Place four dice in front of the group, with each dice showing 5 dots in the top side. Ask the students questions such as: "How many dots are there altogether?" "Record how you worked it out." "If I added one more dice or removed one dice, how many dots would there be?"

Students who solve the problem by multiplication, e.g., 4 × 5 = 20, and are able to add or subtract a group without re-counting or adding understand the distributive property and need to be extended.

Equipment: Animal strips (see Material Master 5–2), multiplication fact cards (see Material Master 6–2)

Using Materials

While the students are not looking, lay down five three-animal strips in an array (rows and columns). Ask the students to work out how many animals there are in total any way they can. Record the strategies that they use with symbols, for example:

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

3 + 3 + 3 + 3 + 3	5 + 5 + 5	10 + 5	6 + 3 + 3 + 3
5 × 3	3 × 5	12 + 3	6 + 6 + 3

Discuss the meaning of the symbols in each expression.

Ask "What should be done to turn the animal array into 7 × 3?" (add two three-animal strips)

Ask "What would the total number of animals be for seven rows of three?" Get the students to explain how they found their answers.

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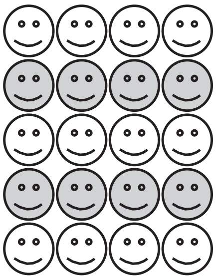
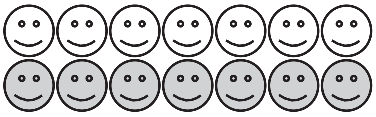
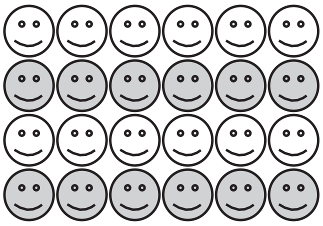
Provide other arrays for the students to see. Record each array as an operation expression as the students suggest them. Suitable arrays might be:



5 × 4



2 × 7



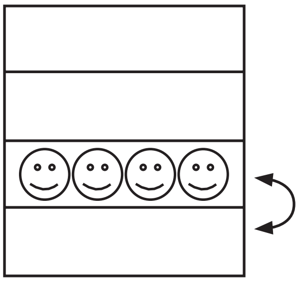
4 × 6

Lay down multiplication cards, say, 6 × 5. Invite the students to form the array and work out the total number of animals involved.

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Form arrays with the animal strips, but turn the cards upside down. Turn one or two cards in the array over and back to allow the students to recognise one of the factors. Ask them what multiplication expression would give the total number of animals.



Record 4 × 4

Allow the students to work out the total number of animals by imaging. Encourage part-whole methods, such as 4 + 4 = 8, 8 + 8 = 16.

Place more animal strips of the same size in the array and ask the students to use the previous operation to work out the next number in the sequence. Similarly, take strips away from the array. Record the operations using multiplication statements: 4 × 4 = 16, so 5 × 4 = 20, so 7 × 4 = 28, and so on.

Suitable arrays might be:

3 × 6 = ☐ so 4 × 6 = ☐ 4 × 5 = ☐ so 6 × 5 = ☐
7 × 2 = ☐ so 6 × 2 = ☐ 6 × 3 = ☐ so 5 × 3 = ☐

Using Number Properties

Provide the students with related multiplication problems. Discuss the meaning of each problem, for example, 10 × 3 = ☐, so 9 × 3 = ☐, as "ten sets of three" and "nine sets of three".

Look for the students to use either knowledge of the fact or part-whole methods to solve each problem and to derive one fact from the other.
For example, 10 × 3 = 30, so 9 × 3 = 27, three less.

Suitable problems are:

2 × 8 = ☐ so 3 × 8 = ☐ 2 × 7 = ☐ so 3 × 7 = ☐
5 × 4 = ☐ so 6 × 4 = ☐ 5 × 6 = ☐ so 6 × 6 = ☐
10 × 4 = ☐ so 9 × 4 = ☐ 10 × 7 = ☐ so 9 × 7 = ☐

Independent Work

Use the animal strips (Material Master 5–2) and the multiplication fact cards (Material Master 6–2) to play the game of Multiplication or Out. (The rules of this game are printed on the material master.)

Pirate Crews

I am learning to solve problems by making equal shares.

Key Mathematical Ideas

- Students need to understand two different contexts for division. These two contexts are *sharing*, as in 12 gold coins shared among four people (Pirate Crews), and a *grouping/measuring* idea, as in 15 marbles put into groups of three (Biscuit Boxes).
- In a sharing problem like Pirate Crews, the amount to share (the number of gold coins) and the number of people (pirates) to share the coins among is known but the number of coins that each pirate would receive is unknown.
- Initially, in learning the sharing-out process, students may deal one coin to each pirate. Some may not recognise the need for equal shares.
- Students move to anticipating the number of coins in each share and use skip counting or adding to check.
- The language used by the teacher and the student while the equipment is manipulated should describe their actions, e.g., "10 coins are shared between 2 pirates, so each pirate has 5 coins." Confident use of this language should be linked to equations, e.g., $10 \div 2 = 5$ and $2 \times 5 = 10$.

Key Mathematical Knowledge

Check that the students can skip count forwards and backwards in twos, fives, and tens.

Diagnostic Snapshot

Ask the students questions such as:

"There are twenty coins to be shared among five pirates; how many coins will each pirate receive?"

Students who solve the problem by multiplication, e.g., $5 \times \square = 20$, by skip counting 5, 10, 15, 20, or by using repeated addition $5 + 5 + 5 + 5 = 20$ need to be extended because they understand that the answers to sharing problems can be found by grouping/measurement.

Equipment: One-dollar coins, paper circles or plates, blank tens frames (see Material Master 4–6).

Using Materials

Problem: "There are two pirates in the crew. They have 10 pieces of gold (which are actually one-dollar coins) to share out. How many coins will each pirate get?"

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Nominate two pirates from the group of students. Ask them to predict the operation and explain their thinking. Many students know that five and five is 10. Link this to fingers, five on one hand and five on the other. Share 10 one-dollar coins between the two pirates to confirm the operation.

Record the result as an equation, $10 \div 2 = 5$. Tell the students that ÷ means "divided by", and that sometimes it means "shared between".

Pose similar problems, getting the students to share out coins to predict the operations. For example:

"Nine coins shared among three pirates"

"12 coins shared among four pirates"

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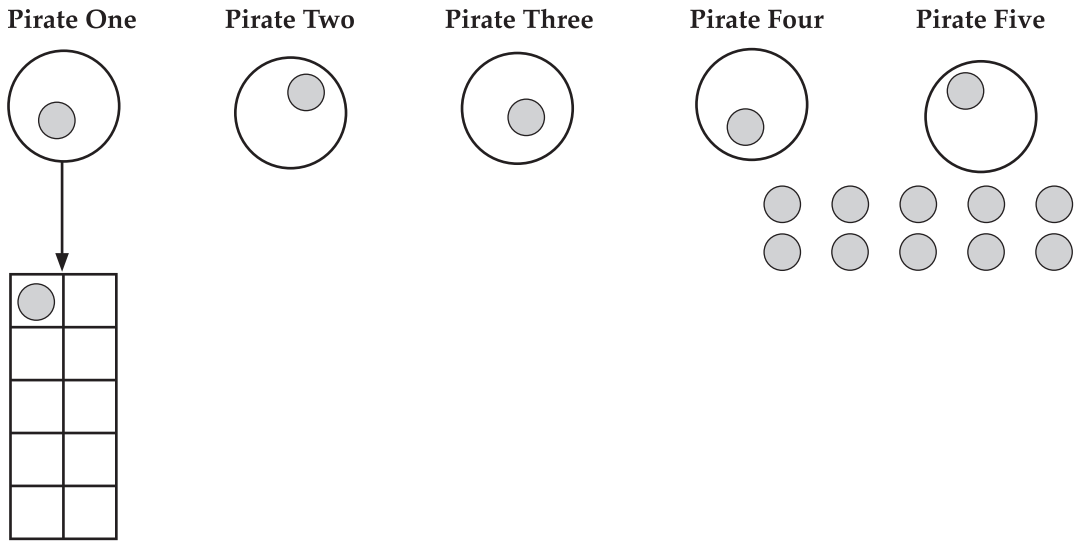
The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Predicting: Set up similar examples using plates or paper circles to collect each pirate's share of the loot. Allow the students to see the number of coins being shared. Arrange the coins on empty tens frames. Encourage prediction by progressive sharing.

For example: "Fifteen coins shared among five pirates"



"You have given one coin to each pirate.

How many coins have been shared?

How many coins are left to share?"

"If you give each pirate one coin more, how much will be left to share?"

"How much will each pirate get in the end?"

Record the operation using symbols, 15 ÷ 5 = 3, and discuss the meaning of each symbol.

Provide other sharing examples, requiring the students to become increasingly adept at predicting the result of sharing and recording each operation as an equation. Fold back to using materials, if needed. Problem examples might be:

"Eighteen coins shared between two pirates" (18 ÷ 2 = 9)

"Twenty coins shared among five pirates" (20 ÷ 5 = 4)

"Sixteen coins shared among four pirates" (16 ÷ 4 = 4)

"Fifteen coins shared among three pirates" (15 ÷ 3 = 5)

Using Number Properties

Provide division problems where the result is easy to predict using known skip-counting sequences. For most students, examples will involve division by twos, threes, fours, fives, and tens. Examples might be:

"Fourteen coins shared between two pirates" (14 ÷ 2 = ☐)

"Fifty coins shared among 10 pirates" (50 ÷ 10 = ☐)

"Twenty-five coins shared among five pirates" (25 ÷ 5 = ☐)

"Twelve coins shared among three pirates" (12 ÷ 3 = ☐)

Biscuit Boxes

I am learning to solve problems by counting the number of groups I have made.

Key Mathematical Ideas

- Students need to understand two different contexts for division. These two contexts are a *sharing* idea, as in 12 gold coins shared between four people (Pirate Crews), and a *grouping/measuring* idea, as in 15 marbles put into groups of three (Biscuit Boxes).
- In a *grouping* problem like Biscuit Boxes, the number of biscuits in each *group* and the total number of biscuits is known, but the number of groups that can be made is unknown.
- Knowing the number in each group allows the students to anticipate repeatedly making groups of that size until all the biscuits have been used (repeated subtraction).
- Doubles knowledge is useful in helping students to anticipate how many groups can be made, e.g., 4 + 4 = 8, 2 groups of 4 is 8.
- Students may skip count or repeatedly add or subtract until they reach the given number. As they skip count, they may track the number of groups on their fingers.
- The language used by the teacher and the student while the equipment is manipulated should describe their actions, e.g., "20 biscuits put into packets of 5 is 4 packets made." Confident use of this language should be linked to the equations, e.g., $20 \div 5 = 4$ or $4 \times 5 = 20$.

Key Mathematical Knowledge

Check that the students can skip count forwards and backwards in twos, fives, and tens.

Diagnostic Snapshot

Ask the students questions such as:

"Twenty pencils are packed five pencils to a box. How many boxes of pencils do we have?"

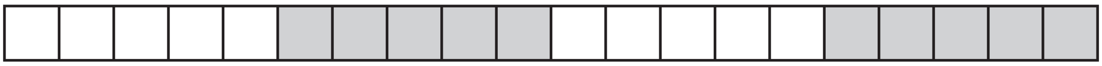
Students who solve the problem by multiplication, e.g., $5 \times \square = 20$, or by using repeated addition, $5 + 5 + 5 + 5 = 20$, need to be extended to deriving from known facts, e.g., $2 \times 5 = 10$, so $4 \times 5 = 20$.

Equipment: Unilink cubes, plastic ice-cream containers, stickies (optional)

Using Materials

Problem: "I'm giving you all a job at the biscuit factory. I want you to pretend that these cubes are biscuits. Make 20 biscuits for me." Encourage the students to make their biscuits in a connected row, with five of each colour until they get to 20:

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*



"Now you have to put the biscuits into packets. You are going to put four biscuits in each packet. How many packets will you be able to make with 20 biscuits?"

Encourage the students to predict the number of packets. Some may do this using the colour pattern of the groups, taking one cube off each group of five to form four groups of four and then taking the remaining cubes from the old "five" groups to make a fifth group of four.

Allow the students to divide the cubes up into fours and record the operation, $20 \div 4 = 5$. Discuss the meaning of the ÷ symbol as "put into sets of".

Pose similar examples using the biscuit factory scenario. With each problem, encourage the students to apply any addition and multiplication fact knowledge they have to predict the result of the operation. Record each operation to consolidate the meaning of the symbols.

Suitable examples might be:

"Twelve biscuits put into packets of three" ($12 \div 3 = 4$)

"Fourteen biscuits put into packets of two" ($14 \div 2 = 7$)

"Thirty biscuits put into packets of five" ($30 \div 5 = 6$)

"Twenty-four biscuits put into packets of six" ($24 \div 6 = 4$)

Using Imaging

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Shielding and Predicting: Set up problems in which the batch of biscuits is made and masked under an ice cream container. Ask the students to predict how many packets of a given number can be formed. Record the operation on a piece of paper fastened to the top of the container, for example, $18 \div 3 = \square$.

Challenge the students to find answers using the number fact knowledge they have. For example:

"If you made two packets of three, how many of the biscuits would you use?"

"What about three packets of three?"

"Can you use this to think ahead and work out how many packets can be made using 18 biscuits?"

$$18 \div 3 = \square$$

Note that the difficult part of a skip-counting strategy is to track how many repeated additions are made. If necessary, unmask the materials and share the cubes to confirm the students' predictions.

Suitable examples might be:

$10 \div 2 = \square$ $15 \div 3 = \square$ $16 \div 4 = \square$ $20 \div 5 = \square$ $60 \div 10 = \square$

Using Number Properties

Ask other division problems, using the biscuit company scenario, for example:

$16 \div 2 = \square$ $24 \div 3 = \square$ $12 \div 4 = \square$ $30 \div 5 = \square$ $90 \div 10 = \square$

The students record the problems as division equations and solve them by applying their strategies and number knowledge.

Twos, Fives, and Tens

I am learning to work out multiplication facts from what I know about twos, fives, and tens.

Key Mathematical Ideas

- Students learn how to derive the two times tables from the doubles, e.g., 7 + 7 = 2 × 7.
- Students learn how to link the 10 times table from their knowledge of the "ty" words, e.g., *6 × 10* is the same as *six*ty.
- Students learn how to derive the five times table from the tens by doubling and halving.

Students who, by the end of the lesson, have demonstrated that they are able to derive the twos, fives, and tens will require opportunities to repeatedly practise the facts until quick recall has been achieved.

Key Mathematical Knowledge

Check that the students know:

- the doubles, e.g., 7 + 7 = 14
- how many tens in a decade, e.g., "How many tens are in 80?"
- the skip-counting sequence for fives and tens, e.g., 5, 10, 15, …

Diagnostic Snapshot

Check whether the students already know the 2, 5, and 10 times tables. If they already have this knowledge and can connect it to equal-sets problems such as "Six sets of five is how many altogether?", they need to be extended.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

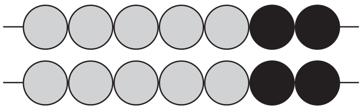
Equipment: Slavonic abacus

Using Materials

Tell the students that they already know a lot of their multiplication (times) tables. This may surprise them, but indicate that you are going to prove it.

Move two strings of seven across on the bead frame. Ask "How many beads have I moved?" "How do you know?"

The students will reply that the answer is 14, since double seven beads have been moved. Tell them that 14 is also the answer to "two times seven". Record the operations as addition and multiplication expressions:



7 + 7 is the same as 2 × 7 (two rows of seven).

Provide similar examples connecting doubles with the two times table and recording the related expressions using symbols. Do this in sequence and ask the students to predict the missing members of the pattern.

5 + 5 is the same as 2 × 5 = 10

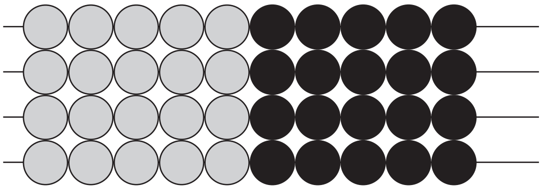
What is 2 × 6?

7 + 7 is the same as 2 × 7 = 14

8 + 8 is the same as 2 × 8 = 16

What is 2 × 9?

Similarly, link the 10 times tables with the students' knowledge of the "-ty" words.

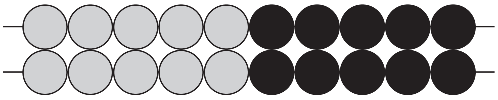


4 × 10 is the same as 40 (forty).

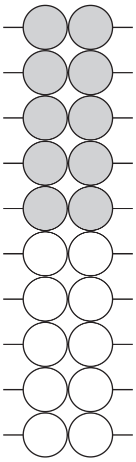
Provide several examples of the "-ty" word link so that the students generalise the idea. Say that 10 could be called "one-ty."

Use the beads to help develop the idea that 10 × 2 has the same answer as 2 × 10:

2 × 10

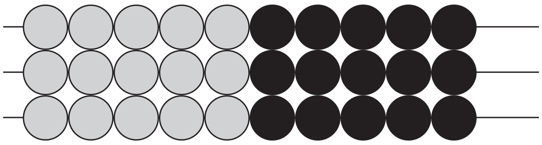


10 × 2



Give other examples, such as 10 × 5 has the same answer as 5 × 10, to generalise the idea.

Connect the five times table to the 10 times table in the following way.



Ask "Here are three tens. How many fives is that?" (six fives)

Record this as: 3 × 10 = 30, so 6 × 5 = 30.

Pose similar related problems, such as:
2 × 10 = 20, so 4 × 5 = 20; and 4 × 10 = 40, so 8 × 5 = 40.

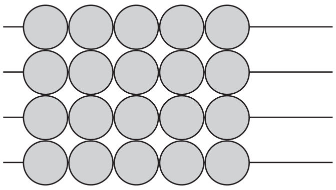
Organise the equations in a pattern and ask the students to derive examples within the pattern:

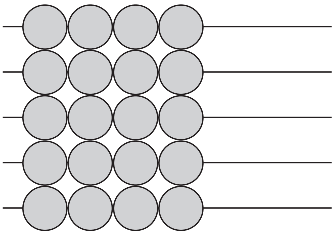
1 × 10 = 10, so 2 × 5 = 10
2 × 10 = 20, so 4 × 5 = 20
3 × 10 = 30, so 6 × 5 = 30
4 × 10 = 40, so 8 × 5 = 40

"What will 3 × 5 be?"

"What will 7 × 5 be?"

Demonstrate that 4 × 5 has the same answer as 5 × 4.





Provide other examples to generalise the idea.
For example: "7 × 5 has the same answer as 5 × 7."

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Using Imaging

Role-playing: Send a student behind a screen with a bead frame. Ask them to move over beads that match the operation you give. For example:

"Tipene, make two times eight for me, please."

Ask the other students to explain what Tipene has done and tell you how many beads have been moved. Record the operations using equations, for example, 2 × 8 = 16. Tipene can be asked to confirm the students' ideas.

Provide many examples to illustrate the connections between the two, five, and ten times tables. Examples might be:

"Make double nine. What is two times nine?"

"Make five times 10. What "-ty" number is that?"

"Make 10 times seven. What "-ty" number has the same answer?"

"Make four times 10. How many fives is that?"

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Using Number Properties

Provide written examples that are outside the number range of the bead frame problems the students have encountered. For example:

2 × 11	2 × 50	2 × 20	2 × 100	2 × 300	2 × 25
10 × 11	12 × 10	20 × 10	100 × 10	15 × 10	25 × 10
20 × 5	5 × 40	12 × 5	5 × 30	18 × 5	13 × 5

The students should be encouraged to realise that the times five problems can be solved by doubling and halving:
12 × 5: double five to make 10, and halve 12 to get 6 × 10 = 60.

Figure It Out activities to reinforce students' knowledge of the 2, 5, and 10 times tables include *Number: Book Two*, Level 2, page 18 (Double Trouble) and *Number Sense: Book One*, Years 7–8, page 9 (Flying Feet).

Learning Experiences to Move Students from Early Additive Part-Whole to Advanced Additive–Early Multiplicative Part-Whole

Required Knowledge

Before attempting to develop their ideas about multiplication and division, check that Early Additive students have the following knowledge.

Key Knowledge	**Questions for Key Knowledge**
• Multiplication facts for the two, five, and ten times tables and the matching division facts	The students can play the game Loopy (see Material Master 6–3) to practise and learn these basic facts. It's important to choose appropriate card sets for each group of students. Loopy is a competitive speed game.

Knowledge to be Developed

While students at the Early Additive stage are developing their ideas about multiplication and division, it is essential that they have the following knowledge (i.e. instant recall) as a long-term goal:

- Learning multiplication basic facts up to 10 x 10 and some division facts; see page 2 of Planning Sheet EA to AA (Mult and Div) www.nzmaths.co.nz/numeracy/Planlinks/MultPlannerEA-AA.pdf and page 2 of Planning Sheet AA to AM (Mult and Div) www.nzmaths.co.nz/numeracy/Planlinks/MultPlannerAA-AM.pdf
- Learning multiplication basic facts with tens, hundreds, and thousands and transferring this to multiplying decades, hundreds, and thousands, e.g., 4×30 is

$$\begin{aligned} 4 \times 3 \text{ tens} &= 12 \text{ tens} \\ &= 120 \end{aligned}$$

$$\begin{aligned} 60 \times 7 &= 7 \times 60 \\ &= 7 \times 6 \text{ tens} \\ &= 42 \text{ tens} \\ &= 420 \end{aligned}$$

Key Ideas

The students at the Early Additive stage are learning to derive their multiplication facts. This means that they can apply a developing range of addition and subtraction strategies to work out unknown multiplication facts from those they already know.

To do so, they need to appreciate key connections between these facts. These connections are:

- 5×4 gives the same answer as 4×5 (commutative property, i.e., changing the order of factors).
- $5 \times 6 = 30$, so $6 \times 6 = 36$ (distributive property, i.e., splitting the factors, e.g., $6 \times 6 = 36$ because $6 \times 5 = 30$ and $6 \times 1 = 6$, the sets of six have been distributed into sets of five and one) and similarly $4 \times 6 = 24$.
- $4 \times 10 = 40$, so $8 \times 5 = 40$ (associative property/proportional adjustment, for example, the double and halve strategy).
- $36 \div 4$ can be answered by solving $4 \times \square = 36$ (reversibility)

A key principle is that **deriving is not an alternative to knowing basic facts.** It's a way to help the students to learn the facts more efficiently and to connect the ideas in an integrated way. Be aware that the students will sometimes apply strategies they have learned through addition inappropriately in multiplication situations. For example, $9 \times 6 \neq 10 \times 5$ (shifting one from six onto nine).

Learning Experiences

Money Changing

I am learning to find out how many ones, tens, hundreds, and thousands are in all of a whole number.

Key Mathematical Ideas

- Understanding place value requires multiplicative thinking because the value of digits is interpreted using both their face value and place, e.g., in the number 5 678, the value of the 6 is 600 because its face value is 6 and its place is in the hundreds column ($6 \times 100 = 600$).
- Place value also involves an important idea: place values are "nested" within other place values, e.g., the number 2 340 contains 234 tens because 100 tens are nested in every thousand and 10 tens are nested in every hundred.

Key Mathematical Knowledge

Check that the students know that:

- "-ty", as in the word sixty, means tens, i.e., sixty is 6 tens.
- moving one place to the left means the place value is ten times more, e.g., 100 is ten tens, 1 000 is ten hundreds, 10 000 is ten thousands, and so on.
- zero acts as a place holder so that a number can be read correctly, e.g., in 700, the zeros hold the ones and tens places, so you read the 7 as 7 hundreds (or seven hundred).

Diagnostic Snapshot

Ask the students questions such as the following to find out if they have the required knowledge:

"You have eighty / forty / sixty dollars in ten dollar notes. How many ten dollar notes is that?"

"How many tens make one hundred? How many hundreds make one thousand?" and so on.

"Denise has $1,000. Deb has $100. How many times richer is Denise than Deb?"

"Read this number: 405." Then 450, 4 050, and so on.

Students who can tell you that there are 47 hundreds and 476 tens in 4 763 already understand nested place value with whole numbers and need to be extended.

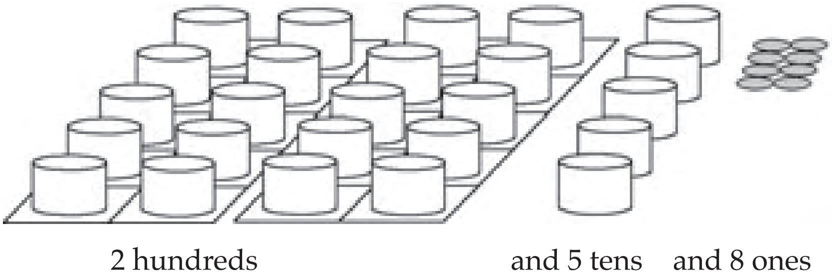
Equipment: Blank tens frames, film canisters, beans, play money

Using Materials

Initially use beans or other single items, such as ice-block sticks or unilink cubes, to model numbers because this is a proportional representation of place value, that is, a canister of ten beans is really ten times more than a single bean and a tens frame holding ten canisters really contains ten times more than a single canister. Numeracy money is a non-proportional representation but is useful because it is easier to manipulate for larger numbers.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Ask the students to make three-digit numbers using beans, e.g., 258 would be modelled as:



Ask the students "nested" place value questions such as:

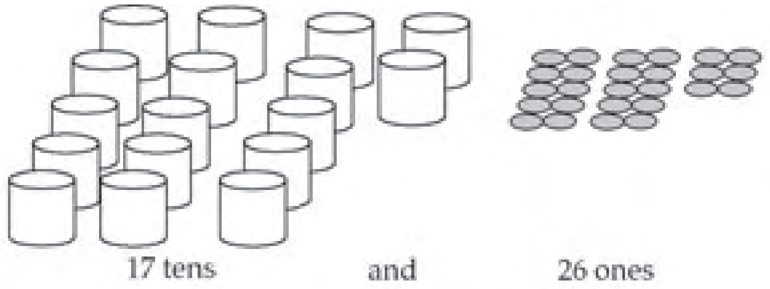
- "How many tens are in our number altogether? Where are all of those tens?" (There are 20 tens in the 2 hundreds, and 5 tens in the tens, that's 25 tens.)
- "How many ones are in our number altogether? Where are all of those ones?" (There are 200 ones in the 2 hundreds, 50 ones in the 5 tens, and 8 ones, that's 258 ones.)

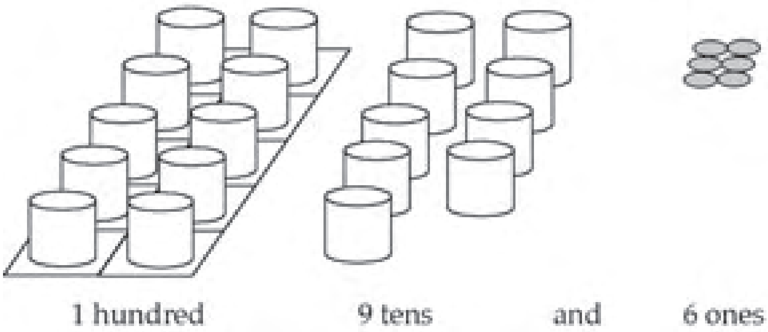
Provide several examples and record students' statements about the same number using words and symbols:

• 3 hundreds and 6 tens and 4 ones	36 tens and 4 ones	364 ones
• 4 hundreds and 7 tens and 3 ones	47 tens and 3 ones	473 ones.

Ask the students to think in reverse by getting them to combine ones and tens, e.g., "Make this number: 17 tens and 26 ones. Find the simplest way you could say this number."

Look for students to combine ten ones as tens, ten tens as hundreds, and so on.





Other examples might be, "Make 29 tens and 13 ones (303). Make 38 tens and 38 ones (418)."

Extend the nested place value concept to thousands and ten thousands, modelling with play money, e.g., "Make 5 671. How many hundreds are there altogether?" (56) "Where are all the hundreds?" "How many tens are there altogether?" (567) "Where are all the tens?" and so on.

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Students may need to build the amounts with hundred dollar notes and ten dollar notes respectively to see these relationships: “Make 18 hundreds, 26 tens, and 35 ones. What is the simplest way to make this number with money?”

Using Imaging

In this phase of the lesson, the focus is on students anticipating the actions on materials, either beans or play money. If students experience difficulty with anticipating, it may be necessary to fold back to manipulating materials or drawing diagrams to model the physical situation. In the process of this imaging phase, recording in word form needs to be connected to the formal arithmetic symbols.

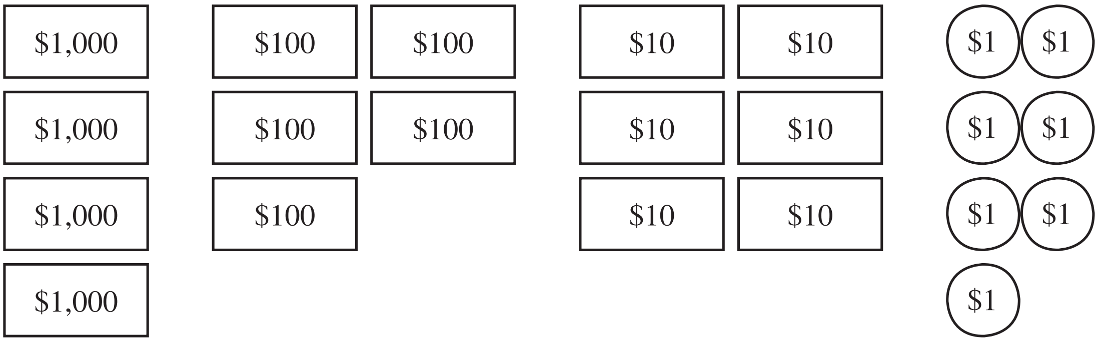
Pose problems such as:

- “Tess has $4,567. She thinks she will feel richer if she changes all her $1,000 and $100 notes for $10 notes. How many $10 notes will she have?”
- “Afisa has this in his money box: seven $100 notes, sixty-seven $10 notes, and twenty-five $1 coins. He goes to the bank. How much money is he depositing?”

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Allow students to record diagrams to work out the problems and record their thinking using words and symbols, e.g.:

“Tess’s money is 4 thousands, 5 hundreds, 6 tens, and 7 ones.



4 thousands	5 hundreds	6 tens	7 ones
40 hundreds	50 tens		
400 tens			

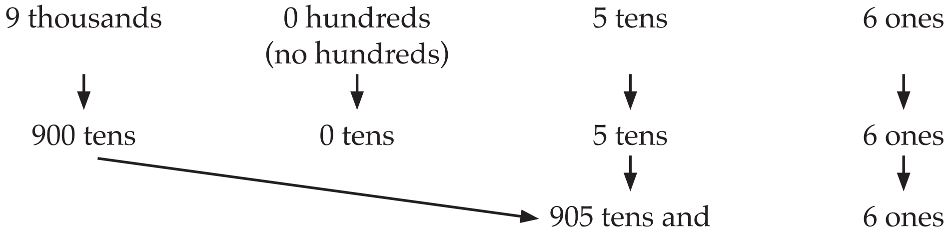
4 567 is 456 tens and 7 ones.”

Using Number Properties

In this phase of the lesson, students need to operate on the numbers rather than relying on images of actions on materials. It is likely that they will need to record information to ease the memory load involved in these problems. Allow them to develop their own systems of recording but encourage them to progressively refine these systems into succinct forms.

Problem: “How many tens in 9 056?” (See the Tess “feeling richer” context above.)

Possible recording:



leading to:

9 056 = 905 tens and 6 ones

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Problem: "How much is 56 hundreds, 63 tens, and 17 ones?"

Possible recording:

$$\begin{tikzcd}
 & \text{56 hundreds} \arrow[dl] \arrow[d] & & \text{63 tens} \arrow[dll] \arrow[d] & \text{17 ones} \arrow[dl] \arrow[d] \\
\text{5 thousands} \arrow[d] & \text{6 hundreds and 6 hundreds} \arrow[d] & & \text{3 tens and 1 ten} \arrow[d] & \text{7 ones} \arrow[d] \\
\text{5 thousands} \arrow[d] & \text{12 hundreds} \arrow[dl] \arrow[d] \arrow[dr] & & \text{4 tens} \arrow[d] & \text{7 ones} \arrow[d] \\
\text{5 thousands and 1 thousand} \arrow[d] & {} \arrow[d] & \text{2 hundreds} & \text{4 tens} \arrow[d] & \text{7 ones} \arrow[d] \\
\text{6 thousands} & \text{2 hundreds} & & \text{4 tens} & \text{7 ones}
\end{tikzcd}$$

leading to:

56 hundreds, 63 tens, and 17 ones equals 6 thousands, 2 hundreds, 4 tens, and 7 ones. 5 600 + 630 + 17 = 6 247

Fun with Fives

I am learning to work out my times six, seven, and eight tables from my times five tables.

Key Mathematical Ideas

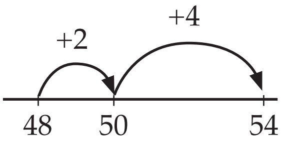
- The students will learn their 6, 7, and 8 times tables through deriving from known five times tables. They will need practice to consolidate this learning.
- The important mathematical idea is the use of the distributive property (splitting factors), e.g., 8 × 7: the sets of seven have been distributed (split) into sets of five and two.

$(8 \times 5) + (8 \times 2)$

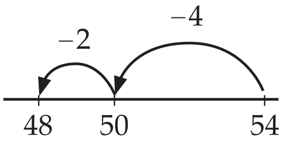
Key Mathematical Knowledge

- Check that the students have quick recall of the 2, 3, and 5 times tables.
- Students will need to have up through ten and back through ten addition and subtraction strategies,

e.g., 48 + 6 as 48 + 2 + 4 = 54



54 − 6 as 54 − 4 − 2 = 48



- Students will need to be able to add two-digit numbers, e.g., answering 7 × 7 through (7 × 5) + (7 × 2) involves adding 35 + 14.

Diagnostic Snapshot

Check whether the students already know the 6, 7, and 8 times tables. Students who already have this knowledge and connect it to equal-sets problems need to go on to Using Number Properties to ensure that they are able to apply the distributive property.

Equipment: Fly flips (see Material Master 4–5)

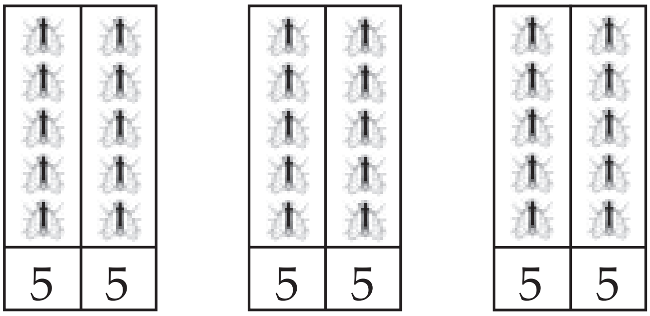
Using Materials

Put down four “five-fly” cards number side up (using fly flips cards).

Ask “How many flies are on these cards altogether?” (20)

“How many flies are on the back?” (none)

Record the operations as $4 \times 5 = 20$. Place more five-fly cards down to illustrate other times five facts. Focus on the relationships between fives and tens by grouping the cards in pairs:



6×5 is the same as 3×10.

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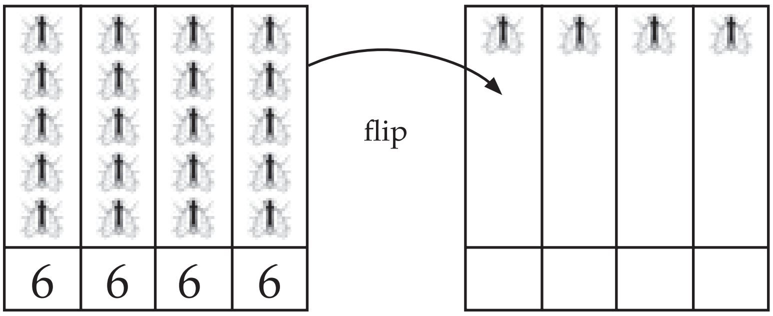
*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Move to examples that show the relationships between multiples of five and other multiples through building on or taking off.

For example, place down four six-fly cards number side up. Ask “The cards show four times six, but you can only see four times five. How many flies is that?” (20) “How many flies do you think will be on the back?” “How do you know?” (four, since there is one fly on the back of each card) “How many flies is that altogether?” (20 and four, 24)



Record the relationship using symbols: $4 \times 6 = (4 \times 5) + (4 \times 1)$

Ask, pointing to the right-hand side of the equation, “What has happened to six on this side?” The students should observe that six has been split into five and one, as in the arrangement on the fly flips cards.

Ask related questions by building onto the array of cards, such as:

6×6 7×6 8×6 10×6 9×6

Pose similar problems using seven-fly and eight-fly cards, for example:

6×7 4×7 8×7 7×7 9×7 4×8 6×8 9×8 7×8 8×8

Note: Your actions as the teacher with the fly flips should be connected to the recording of the equation, e.g., record the groups of five before you flip the flies over. For some students, this may be their first introduction to parentheses (brackets).

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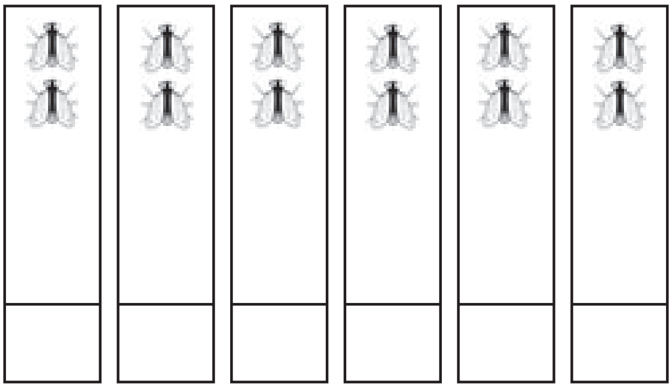
The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Shielding: Put out arrays using five-, six-, seven-, or eight-fly cards placed number side down. For example:



Ask the students to write the multiplication equation for the array and work out the answer. Look for them to use part-whole reasoning and to explain their strategies, for example, "I knew that the cards were seven-fly cards because five and two is seven. There were six lots of five on the front, that's 30, and six twos on the back, that's 12. Thirty and 12 is 42."

Using Number Properties

Providing the students with related problems involving larger numbers will help them to generalise the relationships. Examples might be:

12 × 6　　20 × 6　　14 × 6　　12 × 7　　20 × 7　　11 × 8

Independent Activity

The students can play the game Fly Flip Multiplication (See Material Master 6–4) to consolidate fives groupings.

Multiplying Tens

I am learning to multiply tens, hundreds, thousands, and other tens numbers.

Key Mathematical Idea

Applying knowledge of the basic facts to solve problems that involve multiplying by tens, hundreds, and thousands.

Key Mathematical Knowledge

Students will need to have quick recall of the 2, 3, 5, and 10 times tables, including the reverses and divisions, e.g. 6 × 3 = 18 so 3 × 6 = 18 so 18 ÷ 6 = 3 and 18 ÷ 3 = 6.

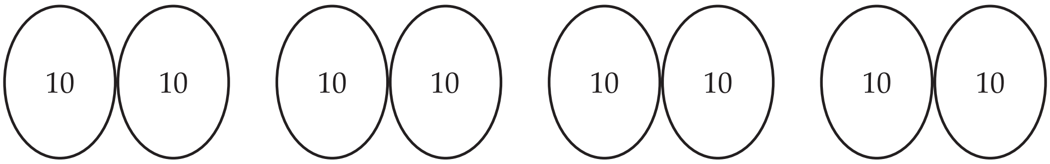
Diagnostic Snapshot

Ask the students questions such as "What is 6 × 300?" Students who already have this knowledge need to go on to Using Number Properties.

Equipment: Beans held as groups of 10 in separate film canisters, 10 canisters to a plastic container (making hundreds), or ice-block sticks/unilink cubes bundled into tens and hundreds or place value blocks or numeracy money, ice cream containers, calculators.

Using Materials

Set up 4 × 20 as a model, using the place value equipment you have chosen:



Ask the students to work out the total number in the collection. Discuss their strategies with a view to connecting $4 \times 2 = 8$ to $4 \times 20 = 80$.

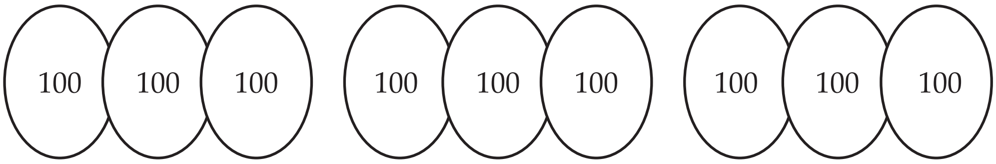
The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Pose other similar problems, modelling them each time with place value materials. Examples could be:

$3 \times 30 = 90$ from $3 \times 3 = 9$ $\quad$ $2 \times 40 = 80$ from $2 \times 4 = 8$

$6 \times 50 = 300$ from $6 \times 5 = 30$ $\quad$ $5 \times 30 = 150$ from $5 \times 3 = 15$

Extend the problems to involve hundreds. Pose the problem 3 × 300.



Link the problem to $3 \times 3 = 9$. Pose similar problems, such as:

$4 \times 200 = 800$ from $4 \times 2 = 8$ or $2 \times 300 = 600$ from $2 \times 3 = 6$

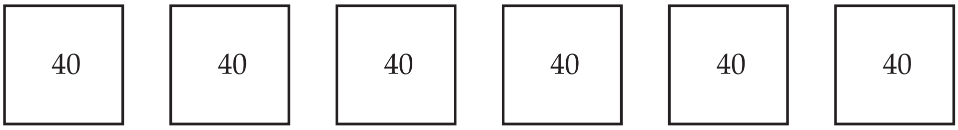
Ask the students to explain the pattern they use to solve these types of problems.

Using Imaging

Shielding: Require the students to image the problems by masking the canisters or stacks with ice cream containers. Label each container with the number involved, using sticky notes. Ask the students to detail the structure of the numbers, for example, "Forty is four tens."

"What is 6 × 40?"

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*



Good examples to use are:

2 × 70 $\quad$ 8 × 50 $\quad$ 4 × 30 $\quad$ 7 × 500 $\quad$ 5 × 400 $\quad$ 6 × 500

Using Number Properties

Use calculators to explore more complex examples so that the students can observe patterns in the answers. Have the students predict answers before they use a calculator. Connected examples might be:

7 × 50 $\quad$ 70 × 5 $\quad$ 500 × 7 $\quad$ 70 × 50 $\quad$ 500 × 70 $\quad$ 700 × 500

60 × 2 $\quad$ 20 × 60 $\quad$ 200 × 6 $\quad$ 60 × 200 $\quad$ 600 × 200 $\quad$ 2 000 × 60

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Look for the students to generalise the number patterns. Some will focus on the role of zero in describing how many digits the number will have. Develop estimation ideas through questions like "I have a number that is a multiple of 10 less than 100 multiplied by a number that is a multiple of 100. How many digits might the answer have?" (one, two, three, or four)

A Little Bit More/A Little Bit Less

I am learning to solve multiplication problems by taking some off or putting some on (compensation).

Key Mathematical Idea

Deriving multiplication facts from known facts, e.g., 5 × 19 would be solved using 5 × 20 and subtracting 5 (compensation).

Key Mathematical Knowledge

- Students will need to have quick recall of the 2, 3, 5, and 10 times tables, including the reverses and divisions, e.g., 8 × 5 = 40 so 5 × 8 = 40 so 40 ÷ 8 = 5.
- Students will need to have up through ten and back through ten addition and subtraction strategies (see the number line examples on page 28)
- Students will need to be able to subtract two-digit numbers, e.g., answering 7 × 8 through (7 × 10) – (7 × 2) involves subtracting: 70 – 14.

Diagnostic Snapshot

Ask the students questions such as "What is 4 × 19?"

Students need to share their strategy. Students who solve the problem by using 4 × 20 and subtracting 4 need to be extended. Ask students who use a place value based strategy, e.g., (4 × 10) + (4 × 9), if they have another way. If not, proceed with the lesson.

Equipment: Unilink cubes, A4-sized masking cards

Using Materials

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

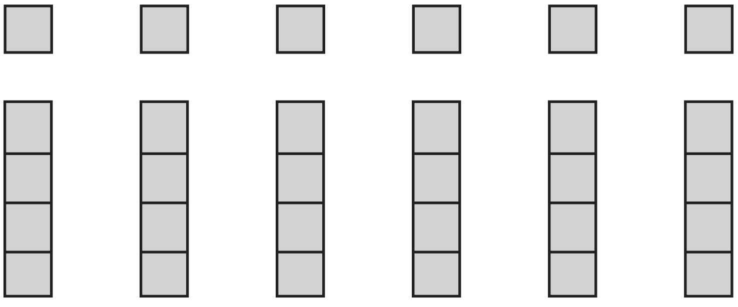
Show the students stacks of five unilink cubes and ask them to confirm the number of cubes in each stack.

Show six stacks of five and ask the students to tell you how many cubes there are in total. Discuss their strategies. These should include gathering pairs of fives to make tens.

Record the operation as an equation, 6 × 5 = 30.

Problem: "What would I have to do to change this into six fours?" (take one cube from each stack) "How many cubes would I have altogether then?" (30 less six is 24)

This can easily be modelled with the stacks of cubes.

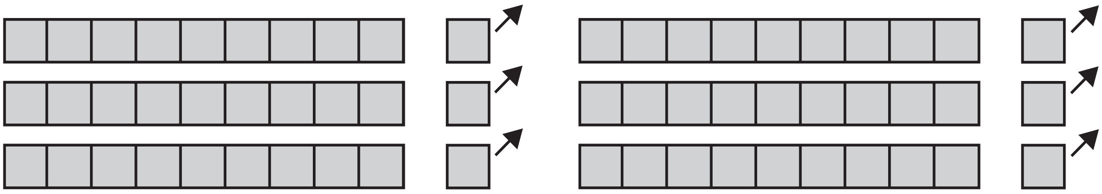


Pose similar problems using groupings of five to derive answers to the four and six times tables. For example:

4 × 5 = 20, so what is 4 × 6? (20 add on 4 is 24) 8 × 5 = 40, so what is 8 × 4? (40 take off 8 is 32)

Use cubes in groups of 10 to show the connections between groupings with 10 and 9 and 11.

Show six stacks of 10 cubes. Ask "How many cubes have I got altogether?" (60) "How could I write this operation?" (6 × 10 = 60) "What would I do to change this to six times nine?" (take one cube from each stack) "How many cubes would I have then?" (60 take off 6 is 54)



*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

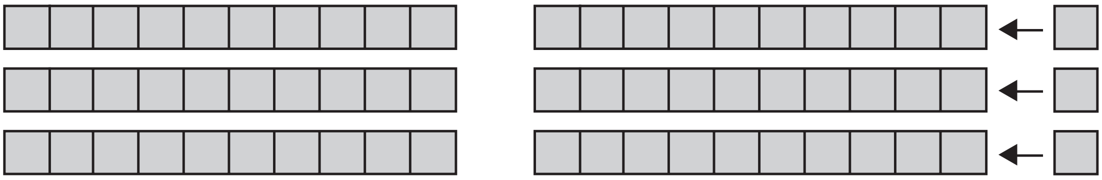
Pose similar problems, such as:

4 × 10 = 40, so what is 4 × 9? 7 × 10 = 70, so what is 7 × 9?

9 × 10 = 90, so what is 9 × 9? 5 × 10 = 50, so what is 5 × 9?

Derive elevens answers from tens answers.

Model the problems with rows of cubes. For example:



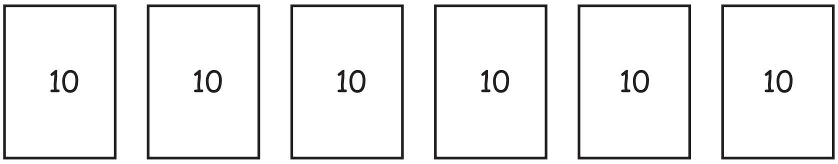
3 × 10 = 30 so what is 3 × 11?

Using Imaging

Shielding: Require the students to image the problems by masking the cubes with large plastic containers. Label each container with the number involved, using stickies.

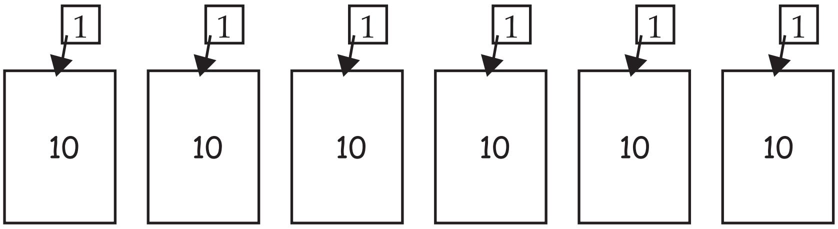
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

For example:



6 × 10 = 60

so what is 6 × 11?



6 × 1 = 6

6 × 10 = 60

(6 × 10 + 6 × 1)

Record each operation using multiplication equations.

Appropriate examples might be:

$7 \times 10 = 70$ so what is 7×11?
$8 \times 5 = 40$ so what is 8×4?
$9 \times 10 = 90$ so what is 9×11?

Using Number Properties

Increase the number size so that the students need to address the number properties rather than relying on images of the materials. Pose the problems as equations, for example:

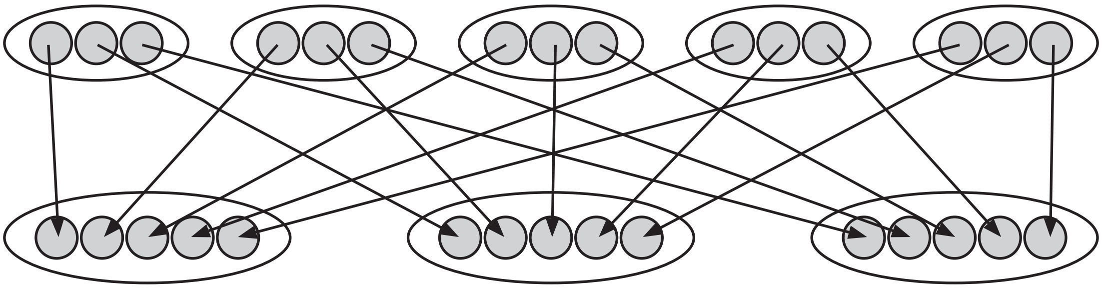
$5 \times 20 = 100$ so what are 5×19 and 5×21?
$4 \times 30 = 120$ so what are 4×28 and 4×31?
$6 \times 50 = 300$ so what are 6×49 and 6×52?
$7 \times 200 = 1\,400$ so what are 7×198 and 7×202?

Turn Abouts

I am learning to change the order of the numbers to make multiplication easier.

Key Mathematical Ideas

- The important mathematical idea is the use of the commutative property (changing order of factors), e.g., 5×4 gives the same answer as 4×5, although in context, the numbers may refer to different things, e.g., 5 bags of 4 oranges is not the same as 4 bags of 5 oranges.
- Students need to understand the transformation that maps a given multiplication onto the commutative equivalent, e.g., with 3×5 and 5×3: 5×3 can be regrouped as 3×5 in this way:



Key Mathematical Knowledge

Students will need to have quick recall of the 2, 3, 5, and 10 times tables, including the reverses and divisions, e.g., $8 \times 5 = 40$, so $5 \times 8 = 40$, so $40 \div 8 = 5$.

Diagnostic Snapshot

Ask the students questions such as:

"Room 4 had to buy 49 tickets at $3 each for the school concert. How much money did they need?"

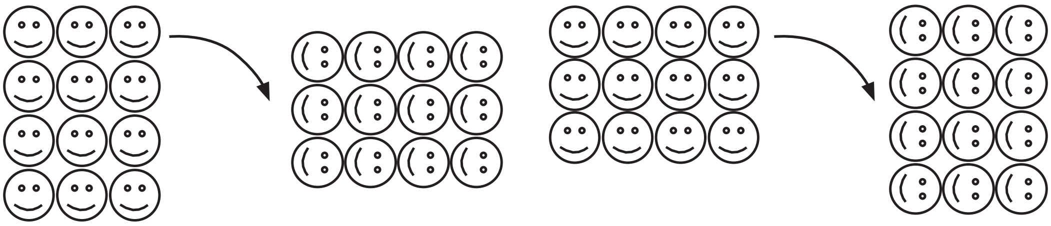
Students who solve the problem by multiplying 3×49 need to be extended because they already understand the commutative property.

Equipment: Animal strips (see Material Master 5–2), unilink cubes, calculators

Using Materials

Ask the students what they would do to model 4 × 3 and 3 × 4 with animal strips. Expect responses like "Get four strips of three animals" and "Get three strips of four."

Tell some of the students to show 4 × 3 and others to show 3 × 4. Compare the total number in each collection and how one model might be mapped onto the other. Some students might comment that 4 × 3 can be rotated to form 3 × 4 and vice versa.



*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

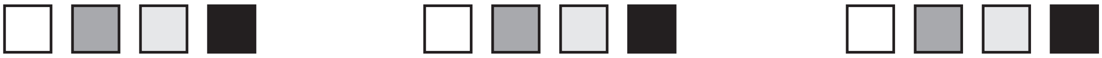
Model other examples with animal strips to establish that the commutative property holds in other cases. Good examples might be:

6 × 2 and 2 × 6　　3 × 7 and 7 × 3　　4 × 8 and 8 × 4

Develop the idea further with cubes by showing the students how four sets of three can be transformed into three sets of four in the following way.

One cube is taken out of each set of three to form sets of four.

This can be done three times.



Challenge the students to use the same process to change:

5 × 2 into 2 × 5　　6 × 3 into 3 × 6　　7 × 4 into 4 × 7

Using Imaging

Record 5 × 8 and 8 × 5 on the board or modelling book. Ask the students to describe what these facts would look like modelled with animal strips and cubes. Expect explanations like "There would be five stacks of eight cubes," and "Eight stacks of five cubes." Focus on how 5 × 8 could be tranformed into 8 × 5 by redistributing the cubes or rotating the columns of the animal strip array to become rows. Ask the students to work out how many animals or cubes would be in each collection and which multiplication would be easier to work out.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Discuss why 8 × 5 is the easiest way to think of the array (students know their 5 times tables, whereas 5 × 8 is comparatively difficult).

Provide other examples in which the choice of multiplication makes a difference to the difficulty of finding the total. Examples might be:

9 × 2 and 2 × 9	(9 sets of 2 and 2 sets of 9: easier to think of double nine)
3 × 6 and 6 × 3	(easier to think of three fives and three more)
4 × 8 and 8 × 4	(easier to think of double eight and then double again)

Record 3 × 100 and 100 × 3 on the board or modelling book. Ask the students if they think the answers to these facts will be the same and ask them to explain why. Use calculators to check whether the commutative property holds.

Give the students other examples, like:　　28 × 2 and 2 × 28

99 × 4 and 4 × 99 (99 sets of 4 and 4 sets of 99)　　6 × 50 and 50 × 6

Using Number Properties

Provide the students with problems involving larger numbers where the choice of calculation makes extreme differences to the difficulty.

Which would you rather work out?

99 × 8 or 8 × 99?	53 × 3 or 3 × 53?	12 × 25 or 25 × 12?
102 × 4 or 4 × 102?	5 × 42 or 42 × 5?	2 134 × 2 or 2 × 2 134?

Independent Activity

To reinforce the commutative property and to learn basic multiplication facts, the students can play Four in a Row Multiplication (see Material Master 6–6).

Long Jumps

I am learning to solve division problems using the repeated addition and multiplication facts I know.

Key Mathematical Ideas

- Students need to understand that *grouping/measuring* division problems involve the repeated subtraction of equivalent groups.
- "Greg the grasshopper is on 24 on the number line. He jumps 3 spaces every time. How many jumps to get back to zero?" This problem can be solved using:
  - skip counting (3, 6, 9, 12, 15, 18, 21) Advanced Counting
  - repeated addition (3 + 3 = 6, 6 + 6 = 12, 12 + 12 = 24) Early Additive
  - repeated subtraction (24 – 3 – 3 – 3 – 3 – 3 – 3 – 3 – 3) Early Additive
  - multiplication facts ($3 \times \square = 24$) Advanced Additive/Early Multiplicative.

  Note that multiplication is the most efficient way to anticipate the number of jumps that Greg will make.
- Students are presented with problems involving remainders and methods of recording for part jumps are explored, e.g., 20 ÷ 3 = 6 r2, $20 \div 3 = 6\frac{2}{3}$, 20 – 3 – 3 – 3 – 3 – 3 – 3 – 2 = 0, or 3 + 3 + 3 + 3 + 3 + 3 + 2 = 20. The choice of how to express remainders depends on the context of the problem.

Key Mathematical Knowledge

Students will need to have quick recall of the 2, 3, 5, and 10 times tables, including the reverses and divisions, e.g., 8 × 5 = 40 so 5 × 8 = 40 so 40 ÷ 8 = 5.

Diagnostic Snapshot

Ask the students questions such as:

"Greg the grasshopper is on 21 on the number line. He jumps 3 spaces every time. How many jumps to get back to zero?" Student who use multiplication or division, i.e., $\square \times 3 = 21$ or $21 \div 3 = \square$, need to be extended.

Equipment: A large class number line 0–100 (see Material Master 4–8), plastic clothes pegs, calculators

Using Materials

Place three different-coloured pegs at the number 24 on a number line. Tell the students that the pegs represent different animals.

“The green peg is a grasshopper which can jump three spaces each time, the yellow peg is a wētā which can jump four spaces each time, and the orange peg is a frog which can jump six spaces each time.” Demonstrate with the pegs how far each animal can jump.

Record this as: green 3 jumps, yellow 4 jumps, orange 6 jumps.

Ask “Each animal wants to jump back to zero. How many jumps will each one take to get there?”

Get the students to predict how many jumps will be required and explain their reasoning. Look for the students to apply their multiplication knowledge, for example, “Two sixes are 12, so four sixes are 24, so with four jumps, the frog can get to zero.”

If necessary, allow the students to jump pegs along the number line to check their predictions. Ask them to record the numbers each animal landed on and how many jumps were taken.

E
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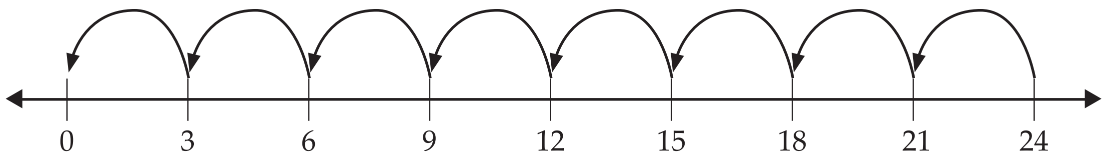
The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

For example, the grasshopper:



Record the results as multiplication, division, and equal addition or subtraction equations.

$8 \times 3 = 24$, $24 \div 3 = 8$, $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 24$, and $24 - 3 - 3 - 3 - 3 - 3 - 3 - 3 - 3 = 0$.

Ask the students where the eight can be found in the equal additions equation, that is, the number of times three is added or taken away.

Compare the equations for the wētā and the frog, $24 \div 4 = 6$ and $24 \div 6 = 4$ respectively. See if the students can identify the common factors.

Pose similar jumping problems such as:

“Three animals are on 20, and the animals jump three, four, and five spaces respectively. How many jumps does each animal take to get to zero?” Note that the three-jumper will go past zero. This raises issues of complete jumps and part jumps. This could be recorded as,

$20 \div 3 = 6$ r2, or $20 \div 3 = 6\frac{2}{3}$, or $3 + 3 + 3 + 3 + 3 + 3 + 2 = 20$, or $20 - 3 - 3 - 3 - 3 - 3 - 3 - 2 = 0$.

Students may also raise the issue of negative numbers as the three-jumper lands below zero with seven jumps.

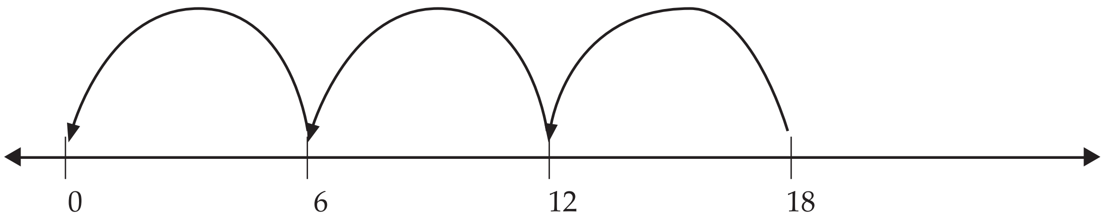
Using Imaging

Pose jumps problems with the number line turned over so that the numbers are not visible. For example, “Possum, Cat, and Dog are on the number 18. Possum jumps three spaces each time, Cat jumps six spaces, and Dog jumps nine. How many jumps will each animal take to reach zero?” Record the information for the students on the board or modelling book.

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Require the students to image the jumps taken and which numbers are landed on. Encourage the application of number facts, particularly the use of reversibility from multiplication facts, such as, “Five threes are 15. One more three is 18, so Possum takes six jumps.”

Record the results as equations and diagrams on the empty number line:



$3 \times 6 = 18$, $18 \div 6 = 3$, $6 + 6 + 6 = 18$, and $18 - 6 - 6 - 6 = 0$

Other examples might be:

"Three animals are on the number 30: one takes three-space jumps, one takes five-space jumps, one takes six-space jumps."

"Four animals are on the number 36: one takes three-space jumps, one takes four-space jumps, one takes five-space jumps, and one takes six-space jumps."

Using Number Properties

Provide the students with examples that involve just the equations, such as:

$15 \div 3 = \square$ $16 \div 4 = \square$ $45 \div 5 = \square$ $70 \div 10 = \square$ $35 \div 7 = \square$

$48 \div 8 = \square$ $42 \div 6 = \square$ $28 \div 4 = \square$ $27 \div 3 = \square$ $32 \div 8 = \square$

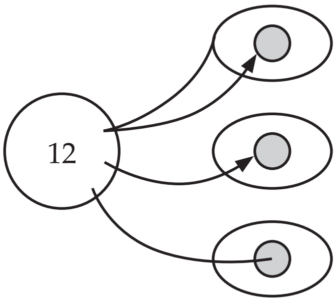
Ask the students to provide a story for each equation.

Goesintas

I am learning to solve "How many sets of ..." and "sharing" division problems using multiplication.

Key Mathematical Ideas

- The problems presented cover both *sharing* and *grouping/measuring* contexts for division.
- Both sharing and grouping/measuring division problems can be solved as multiplication problems where one factor is unknown, e.g., $36 \div 4$ by solving $4 \times \square = 36$ or $\square \times 4 = 36$ (reversibility).
- To gain flexibility in division, the students need to understand the connection between both types of division. Giving a single item to each set in a sharing context requires taking away a set equal to the number of shares, e.g., $12 \div 3 = \square$ as "twelve shared among three".



Giving one to each share is like taking a set of three away from 12. This can be done four times.

$12 \div 3 = \square$ in sharing can be solved as $3 \times \square = 12$, but students tend to think of it as grouping/measuring, $\square \times 3 = 12$.

- Students are presented with division story problems that they are being encouraged to solve using use multiplication (reversibility). Students need to match their actions on the materials with the numbers they record in the equation.

- Students should be using both formats for recording division problems, e.g., $36 \div 4$ and $4\overline{)36}$.

Key Mathematical Knowledge

The students will need to have quick recall of most times tables and their commutative partners, e.g., $4 \times 6 = 24$ and $6 \times 4 = 24$.

Diagnostic Snapshot

Ask the students questions such as:

"Jack has 42 stickers to share out among six people. How many stickers will each person get?"

Students who solve this using multiplication, i.e., $7 \times 6 = 42$, need to be extended.

Equipment: Happy Hundreds arrays (Material Master 6–5) and masking cards, beans, unilink cubes or counters

Using Materials

Expose a row of six faces on the Happy Hundreds array using two masking cards. Pose this problem: "Joneen needs to get 48 doughnuts to feed all the players in her chess club. Doughnuts come in packets of six. How many packets should she get?"

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Record the problem using the equations $\square \times 6 = 48$ and $48 \div 6 = \square$.

Invite the students to predict the answer using whatever mental strategies they have available, such as $5 \times 6 = 30$, so that leaves 18, $6 + 6 + 6 = 18$, $5 + 3 = 8$, therefore $8 \times 6 = 48$.

Their strategies can be verified using the array, for example, by uncovering four more rows, giving five rows of six to show 30 and then successively revealing the other three rows of six.

Pose similar problems that involve sharing the objects in a set into a given number of subsets. For example:

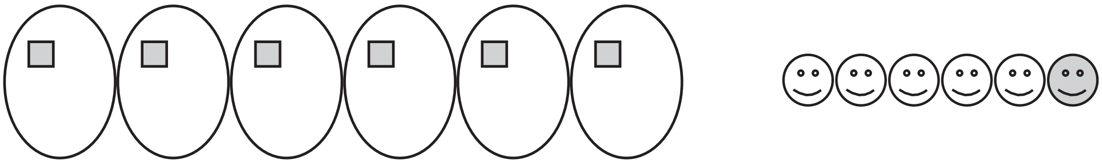
"Geeta has 48 stickers to share out between six people. How many stickers will each person get?"

Record the problem using equations, $6 \times \square = 48$ and $48 \div 6 = \square$.

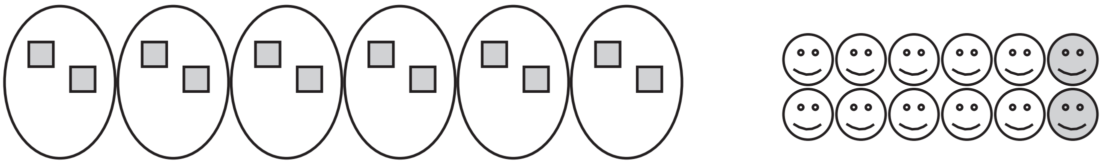
Ask the students to attempt the problem mentally at first, though fold back to Using Materials if necessary. Discuss any strategies the students use.

In responding to students using a dealing strategy, focus on how the results of the dealing could have been anticipated. Use the Happy Hundreds sheet to link the random arrangement of counters to the ordered array that the students have used for multiplication.

"Each person is given one sticker – how many stickers is that?" (six)



"Each person is given two stickers – how many stickers is that?" (12)



Continue until each person has been dealt four stickers, 24 stickers in total.

Ask how many more sixes can be dealt out with the remaining counters.

The students may say that another four stickers can be given to each person as the dealing is halfway through and there are 24 stickers left.

Ask the students to reflect on the relationship between the two problems you have set. The key idea is that both "sets of" and "sharing" division problems can be solved as multiplication problems where one factor is unknown.

For example, $48 \div 6 = \square$ can be solved as $6 \times \square = 48$ or $\square \times 6 = 48$.

Give the students several examples of sharing problems, like:

$35 \div 5 = \square$ (as $5 \times \square = 35$ or $\square \times 5 = 35$)
$28 \div 4 = \square$ (as $4 \times \square = 28$ or $\square \times 4 = 28$)
$27 \div 3 = \square$ (as $3 \times \square = 27$ or $\square \times 3 = 27$)
$42 \div 6 = \square$ (as $6 \times \square = 42$ or $\square \times 6 = 42$)

Use the Happy Hundreds array and counters simultaneously so that the students can see how the dealing operation is mapped by successively revealing rows of faces on the array.

Record the problems as both division and multiplication equations.

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Role-playing: "Henry (student's name) is working out how to share 36 lollies equally among his nine friends. How many lollies should he give each friend?"

Ask the students how the problem could be recorded as an equation,
$36 \div 9 = \square$ or $9 \times \square = 36$.

Ask the students to image what Henry is doing as he shares out the lollies and what that would look like on the Happy Hundreds array. Invite them to provide easy ways for Henry to anticipate the answer, like, "Two nines are 18, so four nines must be 36."

Where necessary, fold back to Using Materials to illustrate the students' explanations if confusion exists.

Pose similar examples in the context of story problems about the number of equivalent sets or sharing objects. Ask the students to describe the actions of a person solving the problems with materials. Record the results as equations.

$45 \div 5 = \square$ (as $\square \times 5 = 45$ or $5 \times \square = 45$)
$36 \div 4 = \square$ (as $\square \times 4 = 36$ or $4 \times \square = 36$)
$24 \div 3 = \square$ (as $\square \times 3 = 24$ or $3 \times \square = 24$)
$49 \div 7 = \square$ (as $\square \times 7 = 49$ or $7 \times \square = 49$)

Using Number Properties

Provide the students with division problems in story form. Ask them to write the corresponding division and multiplication equations before solving them.

$60 \div 5 = \square$ (as $\square \times 5 = 60$ or $5 \times \square = 60$)
$44 \div 4 = \square$ (as $\square \times 4 = 44$ or $4 \times \square = 44$)
$42 \div 3 = \square$ (as $\square \times 3 = 42$ or $3 \times \square = 42$)
$54 \div 6 = \square$ (as $\square \times 6 = 54$ or $6 \times \square = 54$)
$7\overline{)21}$ (as $\square \times 7 = 21$ or $7 \times \square = 21$)
$6\overline{)36}$ (as $\square \times 6 = 36$ or $6 \times \square = 36$)
$8\overline{)24}$ (as $\square \times 8 = 24$ or $8 \times \square = 24$)
$9\overline{)54}$ (as $\square \times 9 = 54$ or $9 \times \square = 54$)

Learning Experiences to Move Students from Advanced Additive–Early Multiplicative Part-Whole to Advanced Multiplicative–Early Proportional Part-Whole

Required Knowledge

Before attempting to develop their ideas about multiplication and division, check that Advanced Additive students have the following knowledge.

Key Knowledge	Questions for Key Knowledge
• Multiplication facts to 10 × 10 and the matching division facts, for example, 54 ÷ 6 • Record the results of multiplication and division using equations	The *Figure It Out* series has three books devoted to basic facts activities. The students might use flash cards to learn the unknown facts. The multiplication and the corresponding division are put on opposite sides of the card, for example, 6 × 7 and 42 ÷ 6. The students can sort the cards so that the multiplication expressions are face up and work through them, checking the answers using the division expressions. Provide written examples for the students to complete that present the facts in a variety of formats, including examples like: $8 \times 4 = \square$, $4 \times \square = 32$, $32 \div 8 = \square$, $\square \div 4 = 8$ $\begin{array}{r}\square\\ 8\overline{)\,32}\end{array}$, $\begin{array}{r}4\\ 8\overline{)\,\square}\end{array}$, $\frac{32}{4} = \square$, $\frac{\square}{8} = 4$ Provide connected examples, like: $7 \times 8 = \square$ so $8 \times 7 = \square$, $2 \times 9 = \square$ so $4 \times 9 = \square$

Knowledge to be Developed

As students' understanding of multiplication and division progresses at this stage, it is essential that the following knowledge is developing simultaneously.

- Recall of basic multiplication facts (see Planning Sheet AA-AM (Mult and Div) www.nzmaths.co.nz/numeracy/PlanLinks/MultPlannerAA-AM.pdf
- Division facts up to 10 times table – see page 2 of Planning Sheet AA to AM (Mult and Div)
- Divisibility rules 2, 3, 5, 9, and 10 – see page 5 of Planning Sheet AA to AM (Mult and Div) and page 70 of this book
- Factors of numbers to 100 including prime numbers – *Book 8: Teaching Number Sense and Algebraic Thinking*, pages 32 and 33

Key Ideas

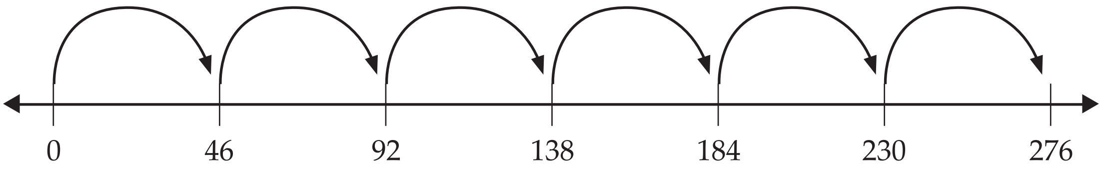
Students at the Advanced Additive stage are developing a broad range of multiplication and division strategies. This requires development of a diverse range of strategies and the transfer of these strategies to division. These strategies are based on transformations to factors that leave the product (the answer) unchanged, e.g., 12 × 27 can be worked out as 2 × 2 × (3 × 27). Splitting 12 into 2 × 2 × 3 leaves the product unchanged.

Important strategies include:

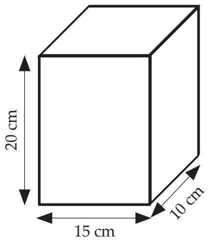
- Compensating from tidy numbers (distributive property), e.g., 6 × 998 = (6 × 1 000) – (6 × 2) = 6 000 – 12 = 5 988 or 76 ÷ 4 using (80 ÷ 4) – (4 ÷ 4) = 20 – 1 = 19.
- Using place value, e.g., 28 × 7 as (20 × 7) + (8 × 7) or 72 ÷ 4 as (40 ÷ 4) + (32 ÷ 4).
- Using reversibility and commutativity (changing order), e.g., 96 ÷ 6 as 6 × ☐ = 96 (reversibility), and changing order, e.g., 17 × ☐ = 102 as ☐ × 17 = 102 (commutativity).
- Using associativity (proportional adjustment), e.g., 4 × 18 as 8 × 9 (doubling and halving factors), or 81 ÷ 3 as (81 ÷ 9) × 3 (tripling and thirding divisor), or 216 ÷ 12 as 108 ÷ 6 as 54 ÷ 3 = 18 (halving both numbers).
- Interpreting division remainders in meaningful contexts, e.g., "378 students travel by bus. Each bus holds 48 students. How many buses are needed?" 7 × 48 = 336 so 378 ÷ 48 = 7 r42. This means that 8 buses are needed.
- Using written working forms or calculators where the numbers are difficult and/or untidy. Preference should be given to working forms that help students to solve the problems in meaningful "chunks" (see example on page 68).

Advanced Additive students also need experience in applying multiplication and division strategies to a range of contexts. These contexts include:

- Multiplicative change and scaling problems such as: "A model car is 20 times smaller than the real car. The real car is 4 metres long. How long is the model?" 4 ÷ 20 = 0.2 metres. The model is 20 centimetres long.
- Lengths and other linear scales, as in measurement problems. For example, "A shoelace is 46 centimetres long. How many shoelaces can be made with 276 centimetres of lacing?"

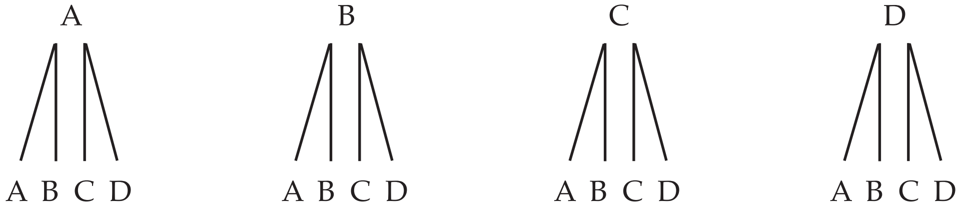


- Areas (arrays) and volumes. "A box measures 20 cm by 15 cm by 10 cm. What is the volume of the box?"



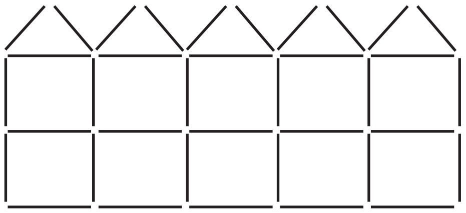
The volume is given by 20 × 15 × 10.

- Cartesian products, as used in finding all the possible outcomes for events in probability. For example, "At the ice cream shop, there are four flavours: apricot, banana, caramel, and date. How many different two-scoop ice creams can you make (assuming that order matters and that two scoops can be the same flavour)?"



The answer is given by 4 × 4 = 16.

- Predicting further members of number patterns, as in algebra problems.
  For example, "How many matches would you need to make 10 houses in this pattern?"



9 + 7 + 7 + 7 + 7 + (7 + 7 + 7 + 7 + 7)

This can be expressed multiplicatively as $9 + (9 \times 7) = 72$ or $(10 \times 7) + 2 = 72$.

- Rate problems involving a multiplicative relationship between two more measures, e.g., "Candice can buy three tops for $54.00. How much will she pay for 12 tops?"

Tops	dollars
3	54
12	☐

Learning Experiences

Sherpa (Tensing)

I am learning to find out what happens to numbers as they are multiplied or divided by ten, one hundred, one thousand, and so on.

Key Mathematical Ideas

- Multiplicative relationships exist between the whole number place values. One hundred is ten times ten, one thousand is ten times one hundred, and so on. One hundred is one thousand divided by ten, ten is one hundred divided by ten, and so on.
- The operations of multiplying and dividing by ten, one hundred, and so on can be represented as a shift of digits to the left or right of the place value columns. For example, in 100 × 73 = 7 300, the digits of the product (7 300) are two places to the left in comparison to the multiplicand (73), with zeros holding the tens and ones places. Note that zeros are not "added" because any number plus zero equals that number, e.g., 56 + 0 = 56.

Key Mathematical Knowledge

Check that the students know the following:

- Multiplication can be interpreted as equal sets, e.g., 7 × 6 = ☐ can mean seven sets of six.
- Division can be seen as sharing a set into equal subsets, e.g., 35 ÷ 7 = ☐ can mean thirty-five shared into seven equal subsets.
- Whole numbers are read by referring to the columns where the digits reside, e.g., in reading 64 078, the 8 resides in the ones column, which in turn sets the 7 in the tens, the 4 in the thousands, and the 6 in the ten thousands, so the number is read "Sixty-four thousand and seventy-eight." Note that in reading large numbers, the three columns in each of the periods thousands, millions, billions, etc., are read collectively, e.g., with 437 029, the thousands period is read as a grouping of three digits, "four hundred and thirty-seven thousand". This idea is commonly represented by place value houses.

Diagnostic Snapshot

Ask the students questions such as the following to find out if they have the required knowledge:

"Show me what 4 × 8 = ☐ looks like with materials" (e.g., interlocking cubes or using happy hundreds arrays).

"Show me what 36 ÷ 9 = ☐ looks like with materials" (e.g., interlocking cubes).

"Read this number: 7 002, 12 509, 436 070, 789 654 321."

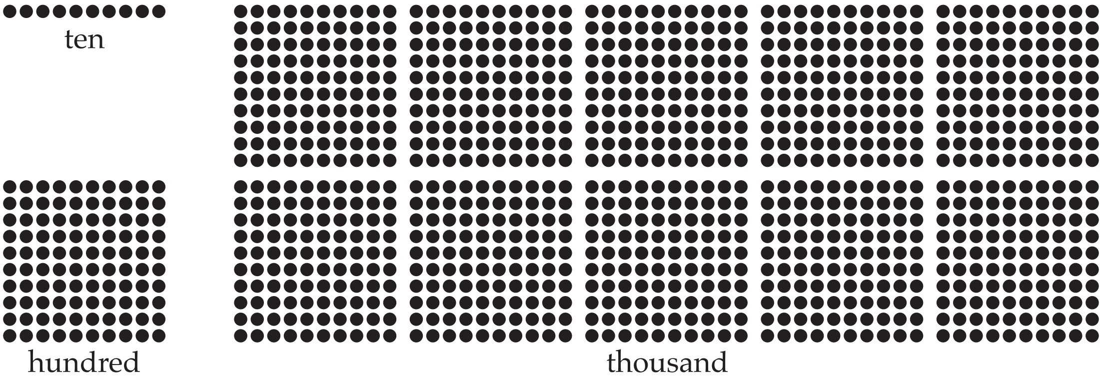
Ask the students questions such as "30 × 43 = ☐ " and "4 300 ÷ 10 = ☐ ". Students who are successful with these questions need to be extended.

Equipment: Copies of large dotty arrays (Material Master 6–9) enlarged to double size onto A3 paper and scissors

Using Materials

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

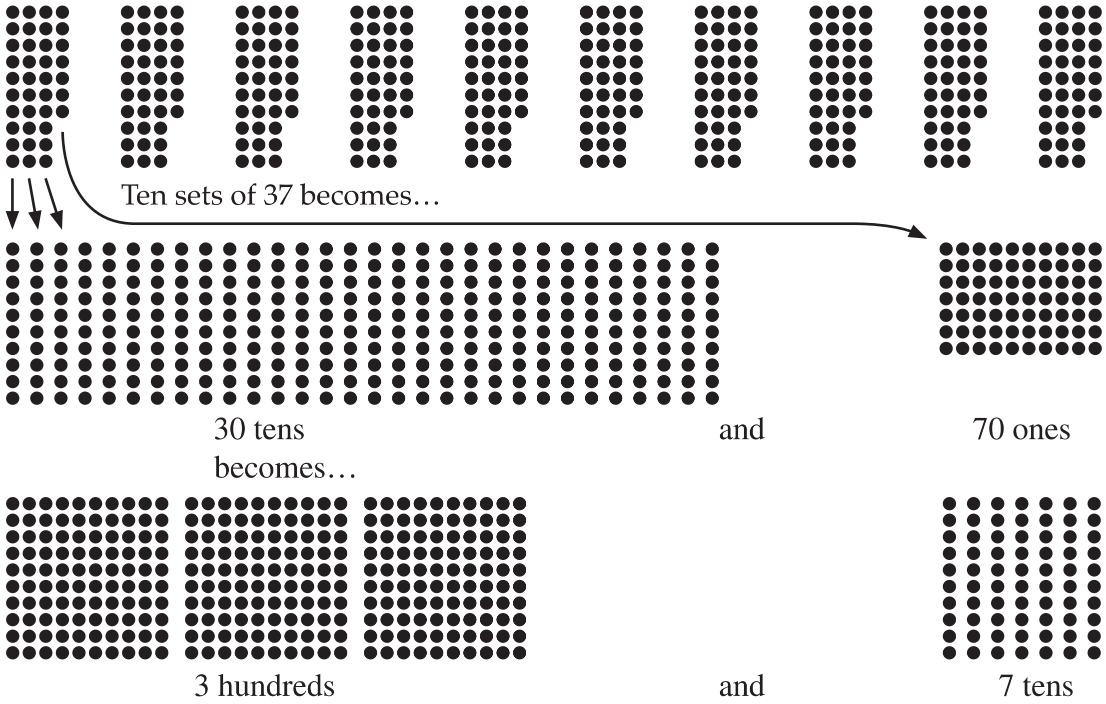
Students unfamiliar with dotty arrays will need to become familiar with the representation of 1, 10, 100, 1 000, and so on as arrays of single dots. This will help them to recognise the relative value of the places.



Ask ten students to make a two-digit number, e.g., 37, using the dotty array pieces. Pose this problem: "Imagine there are ten students and they each have 37 marbles/apples/CDs." Put the sets of 37 into a central space. "How many dots is that altogether?"

Some students are likely to have symbolic algorithms, such as "add a zero", that enable them to get an answer of 370. Examine the actions on materials that explain the use of zero as a place holder. For example:

Ten sets of thirty-seven can be separated into tens and ones.



Using a place value chart, connect 37 with the result of 10 × 37:

Tens thousands	thousands	hundreds	tens	ones
			3	7
		3	7	0

In this way, the students may notice that the digits have shifted one place to the left.

Pose several other problems where ten students make numbers with dotty array parts and look at the combined product. For each example, separate the place values to see what contribution they make to the whole product and write the number and its ten times equivalent on the place value chart.

Further challenge the students by making a two-digit number and posing problems such as, "Imagine that one hundred students had 42 marbles/apples/CDs each. How many would that be in total?" Ask the students how this might be modelled. In these cases, each of the ten students will need to create each number ten times. This is a useful generalisation that shows that ten times ten times of any number is one hundred times that number.

Combine the parts by place value. One hundred sets of 40 becomes 4 thousand, and one hundred sets of 2 becomes 200. On the place value chart, this is displayed as:

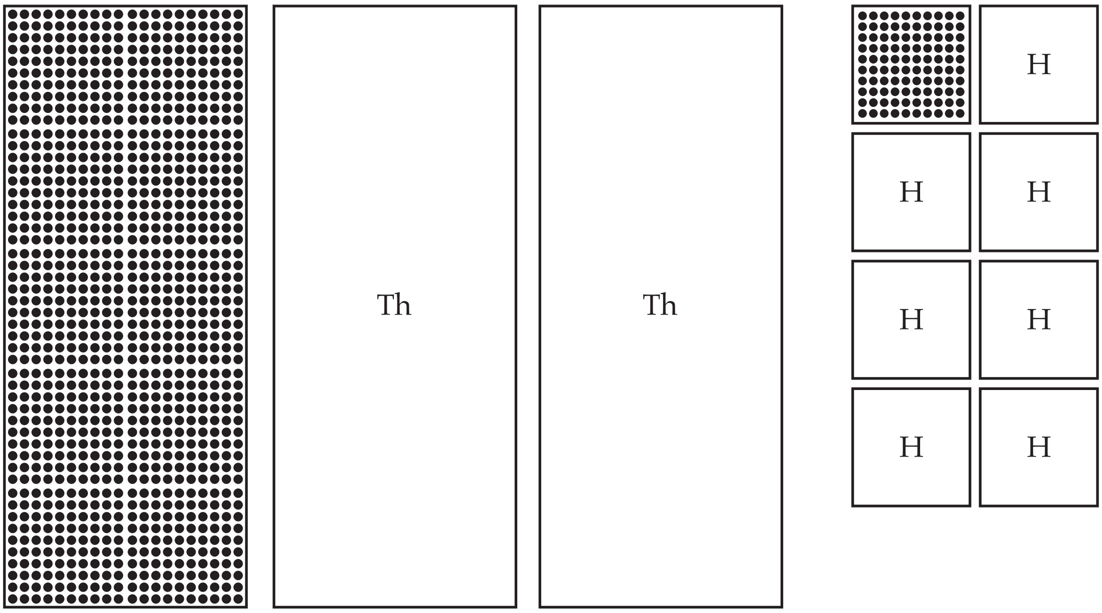
Tens thousands	thousands	hundreds	tens	ones
			4	2
	4	2	0	0

Students may notice that multiplying by one hundred has the symbolic effect of shifting each digit two places to the left.

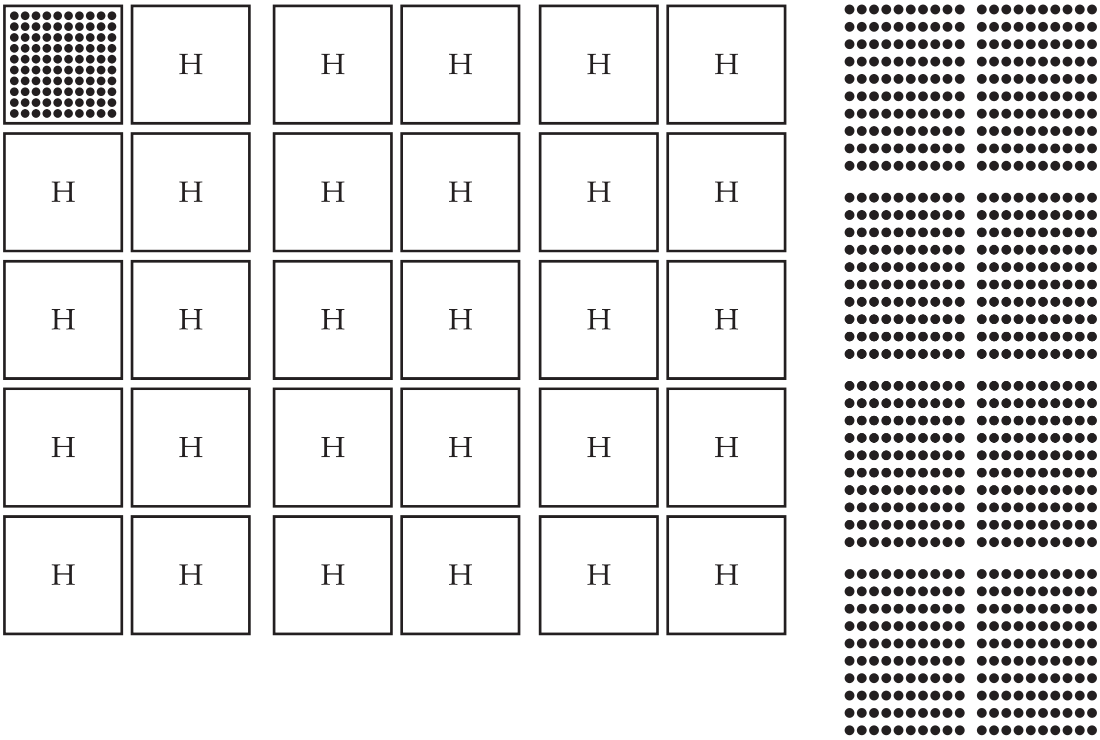
Transfer the focus to dividing by ten and by one hundred. Begin with a four-digit number like 3 800 (zero in the tens and ones places). Make this number with dotty array pieces.

Pose this problem: “I have 3 800 marbles and I am going to share them equally among all ten of you. How many marbles will you get each?”

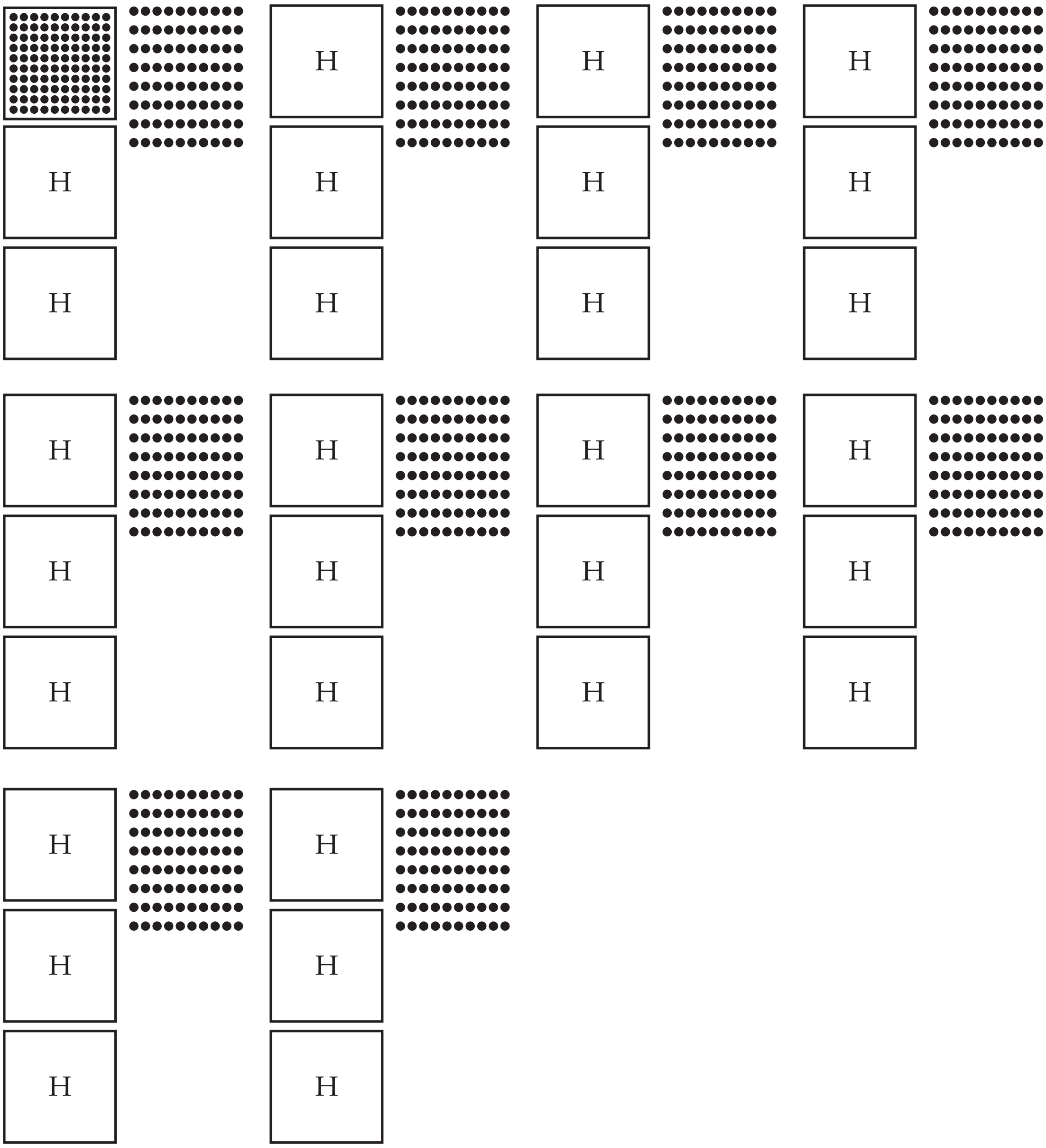
Ask the students to predict the result of the sharing, and then confirm it by modelling with the materials:



3 thousands and 8 hundreds
becomes
30 hundreds and 80 tens



30 hundreds and 80 tens
becomes
ten sets of 3 hundreds and 8 tens



The result of dividing 3 800 by ten can be shown on a place value chart as:

Tens thousands	thousands	hundreds	tens	ones
	3	8	0	0
		3	8	0

The symbolic effect of dividing by ten is to shift the digits of the dividend (3 800) one place to the right. Ask the students to predict what the result would be if they shared 3 800 into one hundred equal sets. Expect them to realise that the shares would be one-tenth of 380, which is 38.

This may need to be acted out by cutting the 3 hundreds in 30 tens and the 8 tens into 80 ones so the tenth shares can be established. Use the place chart to connect 3 800 and the result of 3 800 ÷ 100 = 38. In this case, the symbolic effect is a two-place shift to the right.

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The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Pose several similar multiplication and division problems by ten and one hundred, expecting students to anticipate the results and justify their prediction through explanations about the quantities involved. The dotty arrays can be used to check the explanations where uncertainty exists.

Using Imaging

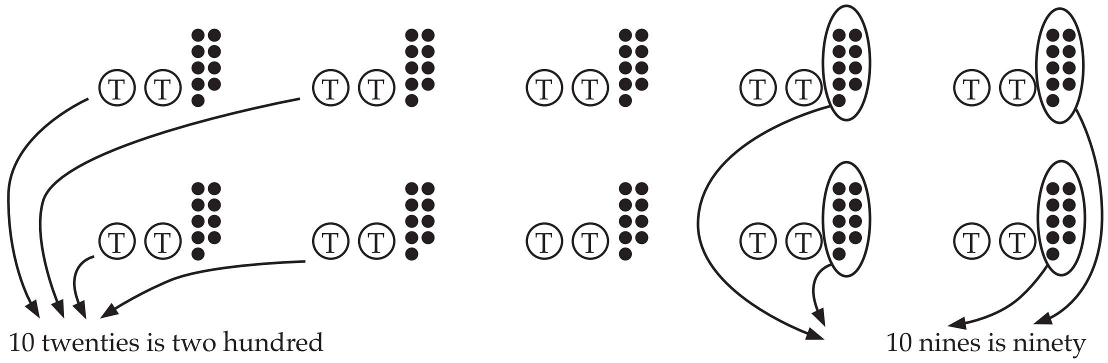
In this phase of the lesson, the focus is on students anticipating the actions on materials, explaining their predictions quantitatively, and drawing diagrams to explain the transformations on the numbers. In the process of this imaging phase, recording in word form needs to be connected to the formal arithmetic symbols.

Pose problems such as:

"Tess pays her ten workers 29 pellowmuffs each. How many pellowmuffs does she pay out altogether?"

Students might use diagrams or words to describe their methods of working out the answer. For example:

"Ten workers get 29 pellowmuffs each like this …"



Altogether, 10 × 29 = 290.

Using Number Properties

Pose problems like these below, expecting the students to reason the answers using place value understanding. The students must be able to justify their answers by explaining what occurs with the quantities involved.

1. 100 sets of 376 (100 × 376 = 37 600)
2. 960 shared among 10 people (960 ÷ 10 = 96)
3. 30 sets of 40 (30 × 40 = 1 200)
4. 4 300 shared among 100 people (4 300 ÷ 100 = 43)
5. 20 sets of 56 (20 × 56 = 1 120)
6. $5,000,000 shared among 1 000 people (5 000 000 ÷ 1 000 = $5,000)

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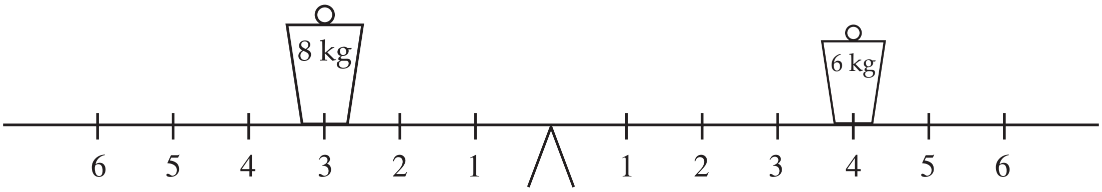
Cut and Paste

I am learning to solve multiplication problems using doubling and halving, thirding and trebling, and other proportional adjustments.

Key Mathematical Ideas

- Students need to understand that problems can be solved through using proportional adjustment (associativity), e.g.:
  - 4 × 18 as 8 × 9 (doubling and halving factors)
  - 3 × 24 as 9 × 8 (tripling and thirding factors).
- Proportional adjustment applies to "product of measures" problems where the factors are transformed multiplicatively but the answer (the product) remains the same. For example:

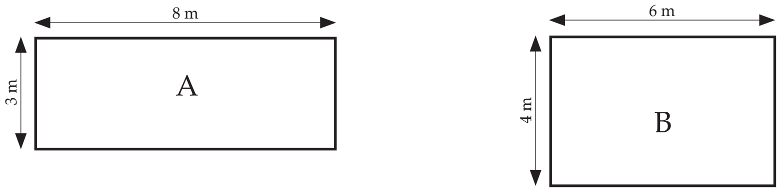
  Balance:



  $3 \times 8 = 24$ and $4 \times 6 = 24$, therefore

  $3 \times 8 = 4 \times 6$, so the scales balance.

  Area:



  Areas A and B are the same, so $3 \times 8 = 4 \times 6$.

- Doubling and halving and thirding and trebling are two examples of a general transformation to the factors that leaves the product unchanged, e.g., 80 × 0.4 = ☐ has the same answer as 8 × 4 = 32. 80 is divided by ten, and 0.4 is multiplied by ten.

Key Mathematical Knowledge

The students will need to have quick recall of all the times tables up to 10 × 10.

Diagnostic Snapshot

Ask the students questions such as "4 × 18". Students need to share their strategy. If a student correctly solves the problems using doubling and halving, ask a problem involving tripling and thirding, e.g., 3 × 27. Students who are also successful with this problem need to be extended. Ask students who use a place value based strategy, e.g., (4 × 10) + (4 × 8), if they have another way. If not, proceed with the lesson.

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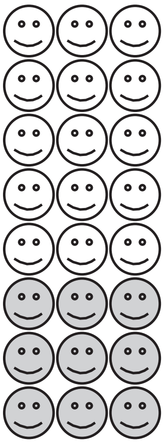
The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

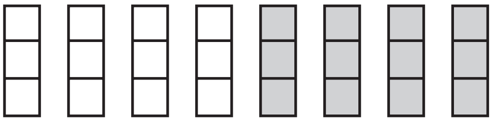
Equipment: Happy Hundreds arrays (see Material Master 6–5), masking cards (half manilla folders), or unilink cubes

Using Materials

Provide each student and yourself with a Happy Hundreds array and two masking cards, or unilink cubes.

Show how eight threes could be displayed on the array, using the masking cards, or with unilink cubes.

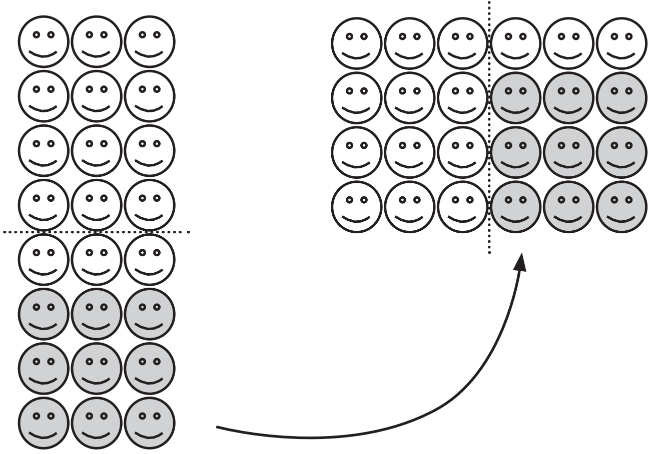




Ask the students to work out how many faces or cubes there are altogether. Discuss the students' strategies. Focus particularly on partitioning, for example, "five threes and three threes" or "double four threes". Record the equation 8 × 3 = 24.

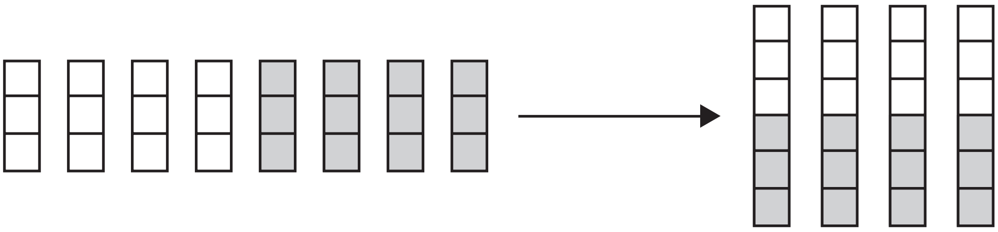
Ask the students to work out how they could change 8 × 3 into 4 × 6.

Students will usually find a solution but, if necessary, show this by partitioning one of the arrays or joining the stacks of cubes. Move the parts as shown (cut and paste).



Record it as:
8 × 3 = 4 × 6

Or



Note that 8 × 3 has the same answer as 4 × 6. Ask the students why they think this is true. The students may notice that each set of factors can be turned into the other by doubling and halving, e.g., with 4 × 6 = 8 × 3, double 4 is 8, half 6 is 3.

Provide similar examples for modelling with the arrays or cubes. These might include:

4 × 5 = 20 2 × 8 = 16 4 × 3 = 12 4 × 10 = 40
2 × 10 = 20 4 × 4 = 16 2 × 6 = 12 8 × 5 = 40

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Using Imaging

Predicting and Generalising: Ask the students to image 3×16 on the Happy Hundreds array or as a cube model. Get them to describe the array/model. Write the expression 3×16 on the whiteboard.

Ask, "How many eights are in three sixteens?" (six eights) "How many cubes is that?" (48)

Record the result as an equation: $3 \times 16 = 6 \times 8$

Pose other problems that can be imaged and solved using doubling and halving. Examples might be:

$18 \times 5 = 9 \times 10$ $3 \times 12 = 6 \times 6$ $14 \times 4 = 7 \times 8$ $24 \times 5 = 12 \times 10$

Say: "Doubling and halving is like multiplying one factor by two and dividing the other factor by two. Will this work if you multiply and divide by three?"

Put up the expression 6×3. Ask "Will this still total 18 if I multiply and divide the factors by three?" The students should note that this transforms 6×3 into 2×9 or 18×1. So trebling and thirding appears to work.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Problem: "Use trebling and thirding to solve 27×3."

Support the students' imaging by modelling the expression with arrays or stacks of cubes and masking. Pose scaffolding questions like:

"Which factor would it make sense to divide by three?" (27)

"How many nines will that be?" (nine nines) "How do you know?" (There are three threes in each nine.)

Pose related examples as given below. Ask the students to use doubling and halving or thirding and trebling.

$2 \times 18 = 4 \times 9 = 6 \times 6$ $24 \times 3 = 12 \times 6 = 8 \times 9$

Ask them to solve 2×36 by multiplying and dividing the factors by four to see if that works as well ($2 \times 36 = 8 \times 9$).

Using Number Properties

Pose multiplication problems with larger numbers that are easy to calculate using strategies such as doubling and halving and trebling and thirding. Larger numbers are hard to visualise, so this will promote generalisation of the number properties. Examples might be:

$6\,482 \times 5 = 3\,241 \times 10 = 32\,410$ $3 \times 18 = 9 \times 6 = 54$

$4 \times 75 = 12 \times 25$ or $2 \times 150 = 300$ $86 \times 50 = 43 \times 100 = 4\,300$

$12 \times 33 = 4 \times 99 = 396$ (using $[4 \times 100] - [4 \times 1]$)

Independent Activity

The students should play the game of Metamorphosis (Material Master 6–7) to consolidate the idea of doubling and halving, trebling and thirding, etc.

Multiplication Smorgasbord

I am learning to solve multiplication problems using a variety of mental strategies.

Key Mathematical Ideas

- Students need to understand that multiplication problems involving multi-digit numbers can be solved through:
  - using *place value*, e.g., 28 × 7 as (20 x 7) + (8 x 7)
  - *rounding and compensating* from tidy numbers, e.g., 5 × 68 as (5 × 70) – (5 × 2)
  - using *proportional adjustment*, e.g., 3 × 27 as 9 × 9. (See also pages 54 and 57.)
  - *splitting factors* multiplicatively, e.g., 12 × 48 = $\square$ can be solved as 3 × 2 × 2 × 48 (note that 3 × 2 × 2 is a factorisation of 12). (See also pages 57 and 76.)
- To solve the problem 9 × 53, some students may need to write down 450 (i.e., 9 × 50) to help keep track of the numbers. Recording parts of a complex calculation as a way to reduce memory load is an important support to advanced multiplicative thinking.
- At this stage, you should model written methods of recording the place value strategy that will be developed further during Paper Power (page 62) and Cross Products (page 67). These can either take a horizontal or vertical form or a grid.

E.g., 5 × 68 = (5 × 60) + (5 × 8)
= 300 + 40
= 340

x	60	8	
5	300 +	40	= 340

Key Mathematical Knowledge

- The students will need to have quick recall of all the times tables up to 10 × 10
- Students will need to be able to *add and subtract efficiently*, e.g., for the problem 6 × 38:
  - a place value approach involves adding 6 × 30 and 6 × 8
  - a rounding and compensation (tidy numbers) approach involves subtracting 6 × 2 from 6 × 40.
- Students will need to connect basic multiplication facts to multiplying by powers of ten, e.g., ten tens are one hundred (10 × 10 = 100), so 8 tens × 3 tens = 24 hundreds (80 × 30 = 2 400)

Diagnostic Snapshot

Ask the students questions such as "Solve 5 × 28 using as many strategies as you can."

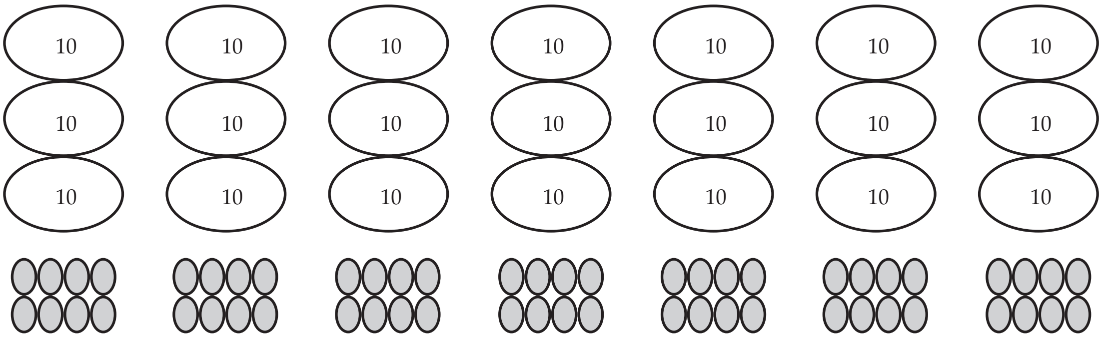
Students who are efficiently using a variety of strategies need to be extended through Using Number Properties and by solving problems with higher powers of ten, e.g., 60 × 37; 40 × 73; 300 × 84.

Equipment: Place value materials (beans, cubes, place value blocks, or numeracy money), containers, stickies (optional)

Using Materials

Problem: "Each container has 38 beans (cubes or dollars). There are seven containers. How many beans are there altogether?"

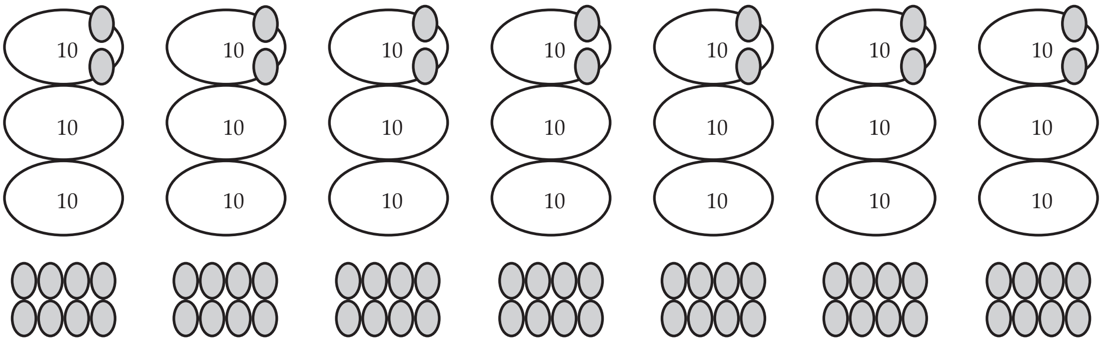
Set up a materials model of the problem:



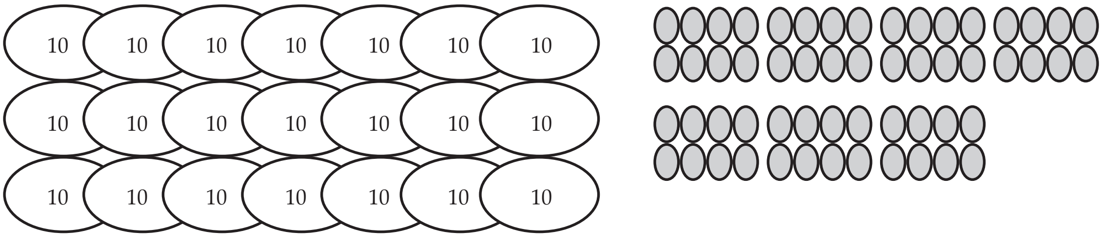
The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Ask the students to think of a way to solve the problem. Get them to explain their strategies. Move place value materials to model each strategy. These strategies might include:

"Adding two to each 38 to make seven forties. That makes 280. Take the seven twos away. That makes 266."



Or: "Seven thirties are 210. Seven eights are 56. Two hundred and ten and 56 is 266."



Provide other examples using materials. Focus on the sharing of strategies and how they relate to actions on the materials. For example: 8 × 26 might be solved as
(8 × 20) + (8 × 6) (using place value)
or (8 × 25) + 8 (using rounding and compensating [tidy numbers])
or (8 × 30) – (8 × 4) (using rounding and compensating [tidy numbers])
or 2 × 26 = 52, 2 × 52 = 104, 2 × 104 = 208 (multiplying and doubling).
3 × 27 might be solved as (3 × 20) + (3 × 7) or (3 × 30) – (3 × 3) or (3 × 25) + (3 × 2) or 9 × 9. Note that the last strategy is proportional adjustment and involves separating each set of 27 into three nines.

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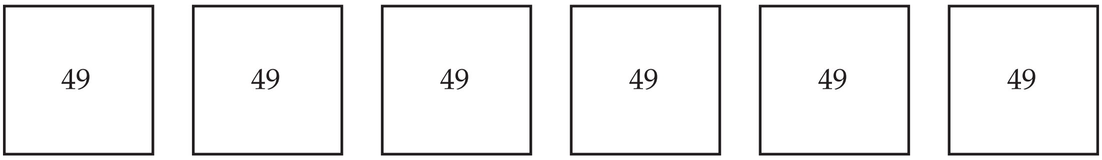
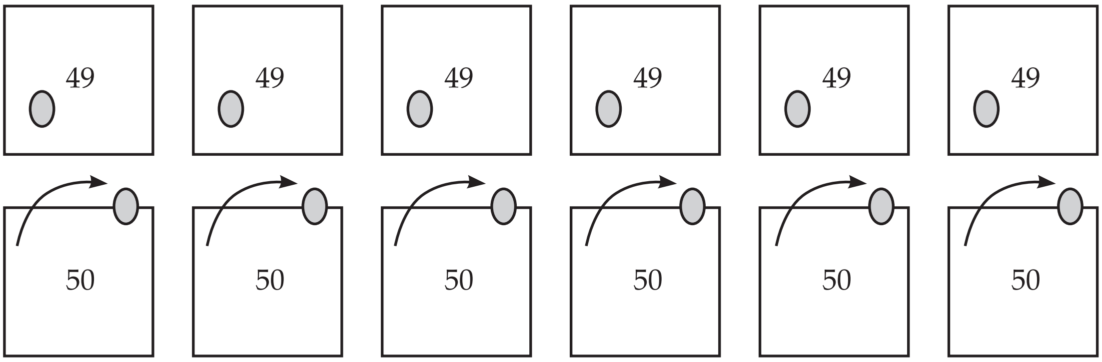
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Shielding: Pose similar problems and mask the place value materials with ice cream containers. Write the numbers involved on stickies, for example, $6 \times 49 = \square$ can be modelled as:



Ask the students to image what is happening to the materials as others explain their strategies. For example, $6 \times 49 = (6 \times 50) - (6 \times 1)$ (rounding and compensating).



Other examples of problems might be:

$5 \times 68 = \square$ $4 \times 97 = \square$ $9 \times 44 = \square$ $3 \times 99 = \square$ $7 \times 26 = \square$

Using Number Properties

Increase the size of the numbers involved, thereby requiring the students to abandon imaging in favour of noticing number properties.

Examples might be:

$8 \times 179 = \square$ $3 \times 66 = \square$ $4 \times 348 = \square$ $9 \times 83 = \square$ $6 \times 78 = \square$

$6 \times 333 = \square$ $486 \times 5 = \square$ $4 \times 275 = \square$ $7 \times 306 = \square$ $5 \times 999 = \square$

Independent Activity

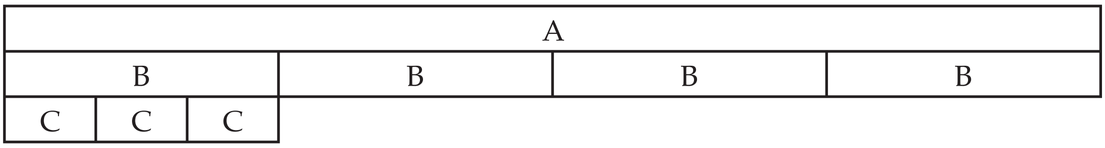
To develop their estimation strategies with multiplication, students should play Multiplication Roundabout (Material Master 6–6).

Proportional Packets

I am learning to solve division problems using how many times one number will go into another.

Key Mathematical Ideas

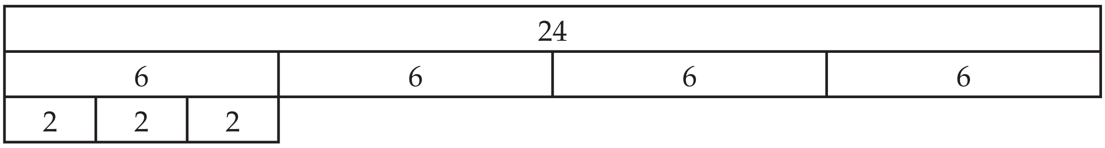
- Solving division problems by calculating "How many Xs go into Y?" is similar to measuring Y with units of X, just like a length might be measured in centimetres or millimetres.
- The size of the measurement unit determines how many times it will fit into whatever is being measured, e.g., consider length A being measured with units B and C:



Four units of B fit into (measure) A (i.e., $4 \times B = A$).

Unit C is one-third the length of unit B, so “three times” as many Cs fit into A ($12 \times C = A$).

- The number of times a number divides into another number works in the same way:



$24 \div 6 = 4$ (How many sixes divide into [measure] twenty-four?)

So $24 \div 2 = \square$ must have an answer “three times” the answer to $24 \div 6 = 4$ (because the units are one-third the size). $24 \div 6 = 4$, so $24 \div 2 = 3 \times 4 = 12$.

- All measurement involves an inverse relationship between the size of the measurement and how many of that measurement fits into the whole,
  e.g., $72 \div 8 = 9$ because $9 \times 8 = 72$
  so $72 \div 4 = 18$ because $18 \times 4 = 72$. Note that the product is constant.

Key Mathematical Knowledge

The students will need to have quick recall of all the multiplication and division facts up to 10 × 10.

Diagnostic Snapshot

Ask the students questions such as “$72 \div 9 = \square$, so what are $72 \div 3 = \square$ and $72 \div 18 = \square$?”

If the student takes a while to solve the problem, it would suggest that they have worked out each question rather than solving them through making connections. Students who are able to explain the proportional idea of division by measurement, e.g., $72 \div 18 = 4$ because $72 \div 9 = 8$ and 18 is twice 9 and 4 is twice 8, need to be extended.

Equipment: Unilink cubes, containers, stickies (optional)

Using Materials

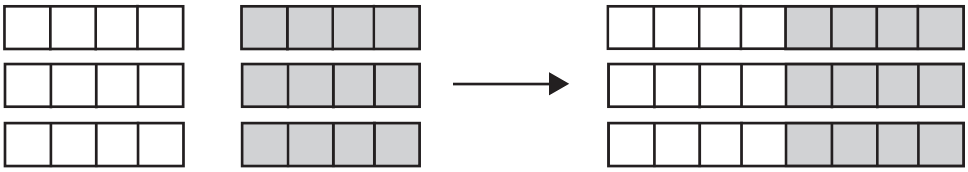
Set up a container with 24 unilink cubes joined in packets of four. Tell the students that there are 24 cubes in total in the container. Take out a packet of four cubes, show it to the students, and then return it to the container. Ask “How many packets of four are in the container?” Discuss the mental strategies used by the students. At this stage, the students should be using reversibility (see page 24), that is, “Six fours are 24, so there are six packets.”

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Ask “How could this problem be written as an equation?” ($24 \div 4 = 6$)

Ask “How could this answer be used to work out how many packets of eight could be made using all the cubes in the container?”

Let the students discuss their strategies. Rather than solving $24 \div 8 = 3$ as a separate problem, look for the students to use proportional adjustment:



Six fours are in 24, and so three eights are in 24.

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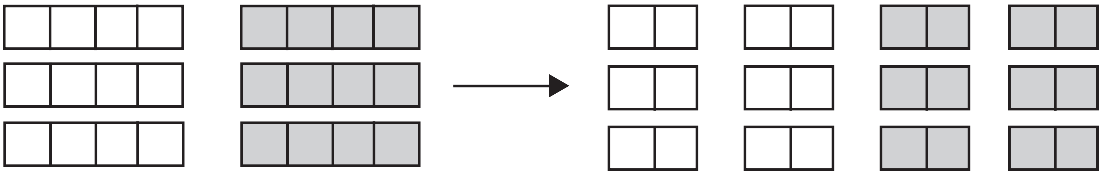
Ask “How can you use this idea to work out how many packets of two could be made with the cubes?”

The students will know the solution using doubling of 12, but look for them to justify it using proportional adjustment:



Six fours are in 24, and so are 12 twos.

Provide other examples in which a division answer is found and proportionality could be used to work out other results. The power of this strategy will become more apparent to the students as the number size increases. Examples might be:

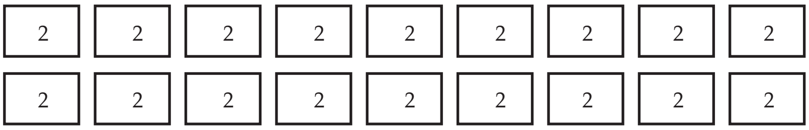
32 ÷ 4 = □, so what are 32 ÷ 2 = □ and 32 ÷ 8 = □?
40 ÷ 5 = □, so what are 40 ÷ 10 = □ and 40 ÷ 20 = □?
18 ÷ 3 = □, so what are 18 ÷ 6 = □ and 18 ÷ 9 = □?

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Role playing: “Suppose that Emily (student’s name) is working at the brick factory. She has a pallet of 36 bricks. How many packets of two bricks can she make?”

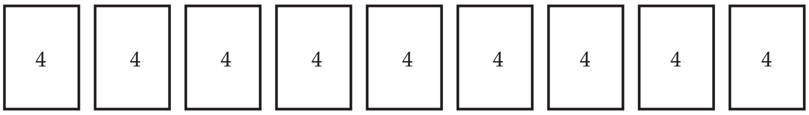
Ask the students to image what the packets look like if laid flat. There will be 18 packets of two bricks. Draw a diagram to assist the imaging.



“Emily thinks that packets of two are too small. How many packets of four can she make with the pallet of bricks?”

Look for the students to realise that two packets of two make a packet of four, and so the number of packets will be halved.

Record this as 36 ÷ 2 = 18, so 36 ÷ 4 = 9



Ask the students to tell you about any patterns they observe in the numbers. Look for ideas such as, “Four is two twos so there are half of 18, that’s nine packets.”

Ask, “How many packets of six bricks could Emily make?”

Record the result as 36 ÷ 2 = 18, so 36 ÷ 6 = 6, noting that because three packets of two make each packet of six, there are one-third as many packets.

Provide further examples in the brick story context, using diagrams to assist the imaging process.

120 ÷ 10 = □, so what are 120 ÷ 5 = □ and 120 ÷ 20 = □?
48 ÷ 6 = □, so what are 48 ÷ 12 = □ and 48 ÷ 3 = □?
72 ÷ 2 = □, so what are 72 ÷ 4 = □ and 72 ÷ 8 = □?

Using Number Properties

Give the students division problems to solve, in a brick packet context, where the use of proportion is a sensible strategy. Encourage them to decide which factor they will divide by to make the problem easier. Examples might be:

170 ÷ 5 = ☐ as 170 ÷ 10 = 17, so 170 ÷ 5 = 34
56 ÷ 4 = ☐ as 56 ÷ 8 = 7, so 56 ÷ 4 = 14
96 ÷ 6 = ☐ as 96 ÷ 3 = 32, so 96 ÷ 6 = 16 (because 3 × 2 = 6) or 96 ÷ 2 = 48, so 96 ÷ 6 = 16
81 ÷ 3 = ☐ as 81 ÷ 9 = 9, so 81 ÷ 3 = 27 (because 3 × 9 = 27)
64 ÷ 4 = ☐ as 64 ÷ 8 = 8, so 64 ÷ 4 = 16, or 64 ÷ 2 = 32, so 64 ÷ 4 = 16

The Royal Cooking Lessons

I am learning to solve division problems by changing them to simpler problems that have the same answer.

Key Mathematical Ideas

- Students need to understand that a complex division problem can be simplified through identifying a common factor, e.g., 32 ÷ 8 simplified to 16 ÷ 4. The common factor of 2 has been used to divide the dividend (32) and the divisor (8).
- Students need to understand that division problems can be solved through using proportional adjustment (often called transforming; see last text box on page 2):
  - 108 ÷ 9 as 36 ÷ 3 (dividing both numbers by the common factor 3 [thirding])
  - 216 ÷ 12 as 108 ÷ 6 as 54 ÷ 3 = 18 (dividing both numbers by the common factor 2 [halving]).

  In this case, both the unit of measure (divisor) and the item being measured (dividend) are being transformed (divided by the same number), e.g.,

A			
B	B	B	B

so

$\frac{1}{2}$ of A			
$\frac{1}{2}$ of B	$\frac{1}{2}$ of B	$\frac{1}{2}$ of B	$\frac{1}{2}$ of B

  If the space being measured is halved and the size of the units is halved, the same number of units fit in:

24			
6	6	6	6

so

12			
3	3	3	3

  24 ÷ 6 = ☐ has the same answer as 12 ÷ 3 = ☐.

- Students need to understand the reversibility connection between division and multiplication, e.g., 40 ÷ 8 as 8 × ☐ = 40. This is what is meant by "How many eights are in forty?"

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Key Mathematical Knowledge

- The students will need to have quick recall of all the multiplication facts up to 10 × 10 and be able to reverse these facts to solve division problems, e.g., 6 × 8 = 48, so 48 ÷ 6 = 8.
- The students will need to be able to find simple common factors given two numbers, e.g., 18 and 24 both have 1, 2, 3, and 6 as common factors.

Diagnostic Snapshot

Ask the students questions such as "At the tyre depot, they need 12 tyres for each truck. How many trucks could they service with 108 tyres?" "How many different ways can you solve this problem?"

Students who correctly solve the problem using proportional adjustment as one of their strategies (halving both dividend [108] and divisor [12]) need to be extended through Using Number Properties.

Equipment: Paper plates or paper circles and counters

Using Materials

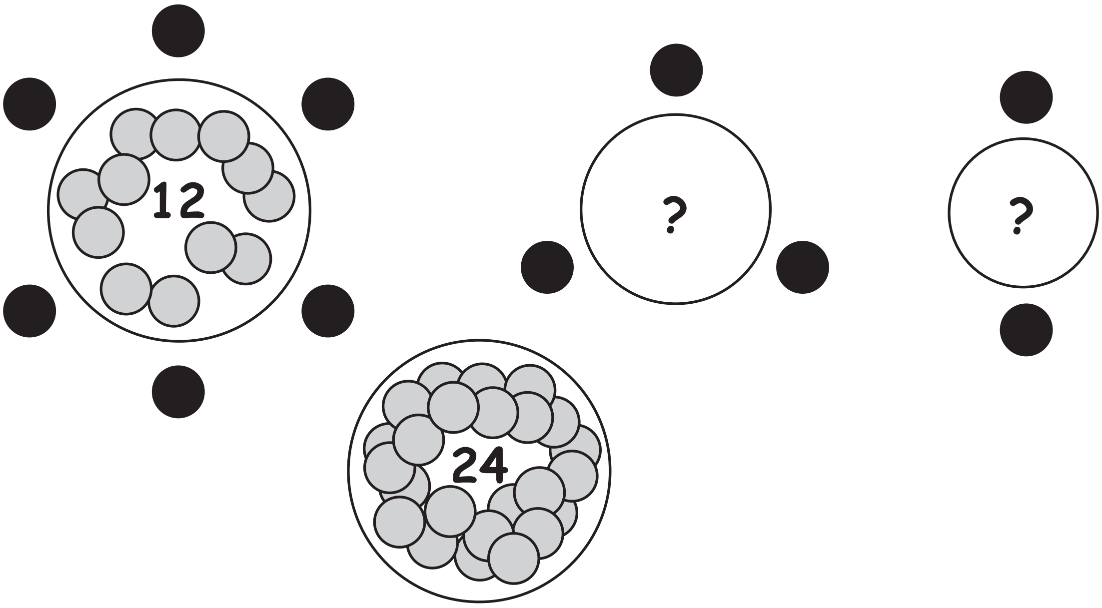
*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Tell the students this story:

"Prince Pikelet, heir to the throne, has been taking creative cooking lessons. He has made a huge batch of spinach-flavoured biscuits that he wants members of the court to try. So he arranges a morning tea.

The servants need to make sure that each guest gets the same share of the biscuits.

At the first table there are 6 seats, so the Prince puts a plate of 12 biscuits on it. There are three other tables. The table arrangement look like this:



How many seats should be at the table with 24 biscuits?"

"How many biscuits should the servants put at each table so that every guest will get an equal share of the biscuits?"

Let the students solve the problems in pairs using materials.

Ask them how their answers could be recorded using equations:

12 ÷ 6 = 2 or 6 × 2 = 12, 4 ÷ 2 = 2 or 2 × 2 = 4, 6 ÷ 3 = 2 or 3 × 2 = 6, 24 ÷ 12 = 2 or 12 × 2 = 24.

Ask them if they notice any patterns in the numbers and to explain how the patterns work. Look for responses such as, "To get the same share means a table with half as many seats gets half as many biscuits."

Provide similar examples using the same biscuit-sharing scenario. Use materials where required to check the predictions made by the students. Suitable examples might be:

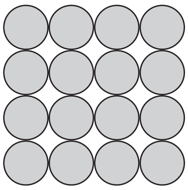
"Sixteen biscuits are at a four-seat table. How many should go to a two-seat table? How many to a six-seat table?"

Get students to model the operations using cubes, counters, or similar materials and record their findings as equations such as $16 \div 4 = 4$ or $4 \times 4 = 16$, $8 \div 2 = 4$ or $2 \times 4 = 8$, $24 \div 6 = 4$ or $6 \times 4 = 24$.

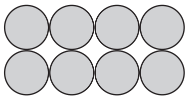
Discuss patterns that the students see in the numbers and why they occur, such as, "If you halve both numbers, the answer is still the same."

Arranging the materials in arrays may help some students recognise the common factor property that is involved.

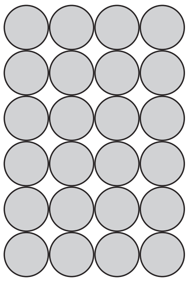
For example:



16 biscuits to 4 people



8 biscuits to 2 people



24 biscuits to 6 people

"Fifteen cookies are at a five-seat table. How many should go to a one-seat table? How many should go to a 10-seat table? How many to a two-seat table?"

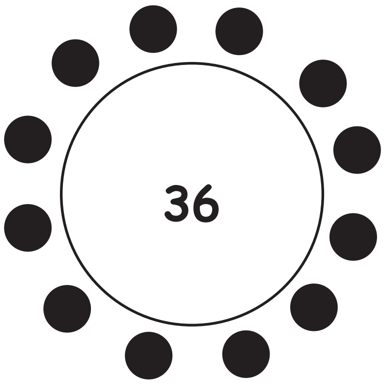
Using Imaging

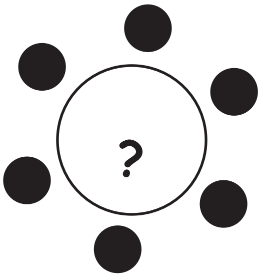
Students should move very quickly through this stage to using number properties.

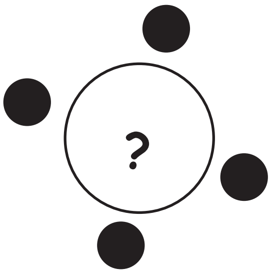
The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Provide the biscuit-sharing problems without materials in a diagrammatic form and look for students to image the sharing process.

"Thirty-six biscuits to 12 people ... How many to 6 people?" "How many to four people?"







Record the solutions as equation patterns to encourage number property noticing.

$36 \div 12 = 3$	$18 \div 6 = 3$	$12 \div 4 = 3$
or $12 \times 3 = 36$	or $6 \times 3 = 18$	or $4 \times 3 = 12$

Other examples might be:

30 biscuits at a 10-seat table.
How many to a ... 5-seat table?
... 2-seat table?
... 20-seat table?

18 biscuits at a 9-seat table.
How many to a ... 3-seat table?
... 6-seat table?
... 18-seat table?

Using Number Properties

Increase the size of numbers involved in the problems so that the students need to use number properties rather than imaging. First examples should involve taking a difficult division and gradually working through progressively simpler problems that have the same answer.

For example:

208 ÷ 8 = ☐ as 104 ÷ 4 = ☐ as 52 ÷ 2 = ☐
216 ÷ 12 = ☐ as 108 ÷ 6 = ☐ as 54 ÷ 3 = ☐

Other problems might be:

484 ÷ 22 = ☐ as 242 ÷ 11 = ☐ (☐ = 22)
140 ÷ 28 = ☐ as 70 ÷ 14 = ☐ as 35 ÷ 7 = ☐
368 ÷ 16 = ☐ as 184 ÷ 8 = ☐ as 92 ÷ 4 = ☐ as 46 ÷ 2 = ☐
390 ÷ 15 = ☐ as 130 ÷ 5 = ☐ as 260 ÷ 10 = ☐
408 ÷ 24 = ☐ as 204 ÷ 12 = ☐ as 102 ÷ 6 = ☐ as 51 ÷ 3 = ☐
324 ÷ 27 = ☐ as 108 ÷ 9 = ☐ as 36 ÷ 3 = ☐

Independent Activity

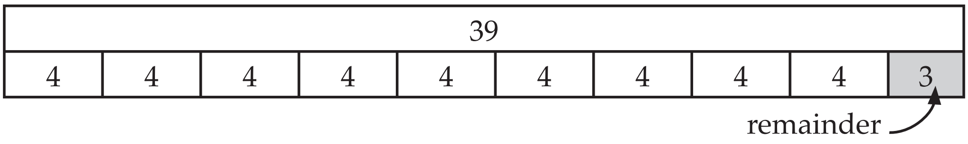
To reinforce using a variety of mental division strategies, students should play Divided Loyalties (see Material Master 6–8).

Remainders

I am learning to solve division problems that have remainders.

Key Mathematical Ideas

- Students need to understand that the solution to a division problem does not always provide a whole number answer.
- The amount left (the remainder) can be expressed as a whole number, e.g., 5 left over, or as a fraction, $\frac{1}{4}$, or as a decimal, 0.25.
- Fraction and decimal remainders refer to the divisor (number dividing by), e.g.,



36 ÷ 4 = 9

so 39 ÷ 4 = 9 $\frac{3}{4}$ or 9.75.

In this case, $\frac{3}{4}$ means $\frac{3}{4}$ of a measure of four (i.e., $\frac{3}{4}$ of 4 is 3) and 0.75 means $\frac{75}{100}$ of a measure of four (i.e., $\frac{75}{100}$ of 4 is 3). Both $\frac{3}{4}$ and 0.75 refer to part of the divisor, four.

- The decimal remainder can be interpreted as both a whole number and as a fraction of a set, e.g., Lisa has 15 metres of cloth. It takes 4 metres to make a tracksuit. How many tracksuits can Lisa make? 15 ÷ 4 = 3.75. The 0.75 can be interpreted as $\frac{3}{4}$ of 4 metres, which is 3 metres, or as $\frac{3}{4}$ of a tracksuit.
- Students should be using more than one format for recording division problems, e.g., 36 ÷ 4 and $4\overline{)36}$.
- How the remainder is expressed is determined by the context in which the problem has been set, e.g., 15 toys shared among 4 people (3 r3), $15 shared among 4 people ($3.75), 15 pies shared among 4 people ($3\frac{3}{4}$). NB: Sometimes the remainder is discarded or rounded,
  e.g., "A piece of rope is 25 metre long. How many 4 metre jump ropes can be made?" (Discarded) "The ferry can hold 8 cars. How many trips will it have to make to carry 25 cars across the river?" (Rounded)

Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 x 10 and are able to reverse these facts to solve division problems, e.g., 9 x 8 = 72, so 72 ÷ 8 = 9.

Diagnostic Snapshot

Ask the students questions such as:

- "The 29 students from Room 9 need to get into four equal lines to play Dodgeball. Is this possible, and how many will be in each line?"
- "The students from Room 2 were cleaning their aquarium; they needed to empty out 60 litres of water. If they used an 8 litre bucket, how many bucketfuls would that be?"

Students who are able to solve problems that involve a whole number or fraction remainder need to proceed to Using Number Properties. Check the recording of their solutions e.g., 7 r4, $7\frac{1}{2}$, or $7\frac{4}{8}$.

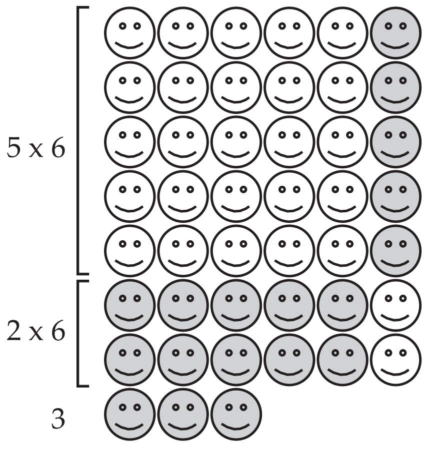
Equipment: Unilink cubes, beans or counters, place value materials, Happy Hundreds arrays (see Material Master 6–5), calculators

Using Materials

Problem: "Mere (student's name) has gathered 45 eggs for the gala. She wants to put them into cartons of six. How many cartons can she fill? Will any eggs be left over?"

Allow the students to use materials and any mental strategies that they wish to solve the problem.
Discuss their strategies.

Look for efficient methods and use the materials to illustrate them. For example, using $\square \times 6 = 45$ (reversibility): five times six is 30, leaving 15 to be shared (distributive property), and six twos are 12, leaving three eggs over.



The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

This could be illustrated on the Happy Hundreds array, as shown above.

Discuss how the remainder of three could be recorded:

45 ÷ 6 = 7 r3 (meaning seven and three left over), or

45 ÷ 6 = $7\frac{3}{6}$ or $7\frac{1}{2}$ (meaning seven and one-half of a whole carton remains).

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Invite the students to key the division into a calculator. This will give the result 7.5. Ask the students how this relates to the other ways of recording the remainder, that is, 0.5 is the decimal for one-half.

Ask "Which way of recording the remainder is the best in solving Mere's problem?" In this case, $7\frac{1}{2}$ gives the exact number of cartons Mere can make, but 7 r3 is more useful because it tells how many cartons can be made and how many eggs are left.

Pose a different problem to help the students to appreciate the usefulness of the "left over" way of recording the remainder.

Problem: "Tony (student's name) has $43. Videotapes cost $5 each. What is the largest number of tapes Tony can buy, and how much money will he have left over?"

Allow the students to solve the problem using materials, if necessary, and discuss their strategies. In this case, recording the division as 43 ÷ 5 = 8 r3 gives both the number of tapes bought (8) and the remaining money ($3).

Provide other examples for the students to solve using materials and ask them to express the remainder in ways that match the demands of the problem. For example:

(1) "Henk has 55 stamps. He needs six stamps to post each parcel. How many parcels can he post? How many stamps will he have left?" (55 ÷ 6 = 9 r1)

(2) "Salena can wash a car in 8 minutes at the fundraising carwash. How many cars can she wash every hour?" ($60 \div 8 = 7\frac{4}{8}$ or $7\frac{1}{2}$)

(3) "Trent has used 26 metres of rope to make eight skipping ropes. Each one is the same length. How long is each skipping rope? Express the fraction part of your answer as a decimal." (26 ÷ 8 = 3.25 m)

Using Imaging

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Role-playing: Provide similar examples in which a student from the class goes away from view to solve the problem with materials. The other students image what he or she is doing and what the remainder will be.

The Happy Hundreds array or place value materials like beans or play money are the easiest materials to visualise with. This does depend on the size of the numbers.

Suitable examples might be:

(1) "Charlie's Chocolates come in boxes of four. He has 39 chocolates to put into boxes. How many boxes can he fill?" (9 boxes, 3 chocolates left over)

(2) "Coach Casie needs seven people to make each netball team. She has 46 players. How many teams can she make? How many players will be reserves?" (6 teams; 4 reserves)

(3) "Romi ran six lengths of the court in 76 seconds. What was his average time per length in seconds? Express the fraction part of your answer as a decimal." ($12.6\overline{6}$ seconds)

Using Number Properties

Provide the students with division problems written as equations. Ask them to solve each problem, express the quotient and remainder in one of the three ways given, and then make up a word story to match their recording, for example, 34 ÷ 6 = 5 r4.

Word story: "Thirty-four people wanted to play volleyball. There were six players in each team. How many teams were there, and how many players were reserves?" (5 teams, 4 reserves)

Useful equations to use as examples are:

84 ÷ 9 = ☐ 47 ÷ 3 = ☐ 35 ÷ 4 = ☐ 51 ÷ 6 = ☐ 472 ÷ 5 = ☐

Use a calculator to solve more difficult division problems. Ask the students to interpret the decimal remainder as both a whole number and as a fraction of a set.

Paper Power

I am learning to work out multiplication and division problems using written working forms.

Key Mathematical Ideas

Multiplication

- Students need to understand and use an efficient written method for multiplication and division. Written methods are used when the problem is too difficult for mental methods and a machine is unavailable.
- Links need to be made to previous methods of recording e.g., for 5 × 68:

from	5 × 60	300	to	68		to	$^{4}68$
(using	5 × 8	+ 40		x 5			x 5
place		340		40	5 × 8		340
value)				300	5 × 60		
				340			

- Students are introduced to a short method of multiplication that involves the recording of the regrouped "ones" above the tens value (see the third method above). This is a complex concept and the connections between the equipment, the action, language, and the recording need to be made explicit. Students will often naturally use place value strategies that can be "captured" in written working form.

Division

- Links need to be made to previous methods of recording, e.g., from 36 ÷ 4 and $4\overline{)36}$ to 92 ÷ 4 as $4\overline{)92}$:

  $4\overline{)92}$ can be recorded as $\begin{array}{r} 20 + \ \ 3 \\ 4\overline{)80 + 12} \end{array}$ (partitioning), then moving to $\begin{array}{r} 2\ 3 \\ 4\overline{)9^{1}2} \end{array}$ (regrouped)
- Students are introduced to a short method of division that involves the recording of the regrouped tens in front of the ones. This value is now read as ones, e.g., 12 is 12 ones.
- Research suggests that recording methods that allow students to "chunk" their calculations into manageable parts are more reliable than shortened algorithms, e.g., "How many fours are in 937?" or $4\overline{)937}$ might be solved in chunks as:

$$\begin{array}{r|l} 4\overline{)937} & \\ -\,400 & 100 \\ \hline 537 & \\ -\,400 & 100 \\ \hline 137 & \\ -\,80 & \ \ 20 \\ \hline 57 & \\ -\,40 & \ \ 10 \\ \hline 17 & \\ -\,16 & \ \ \ \ 4 \\ \hline 1 & 234\text{ r1} \end{array}$$

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Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 x 10 and are able to reverse these facts to solve division problems, e.g., 9 x 8 = 72, so 72 ÷ 8 = 9.

Diagnostic Snapshot

Ask the students to solve problems such as the following using a written division method: "The gardener had 737 trees to plant into three rows. How many trees will he put into each row?"

Ask the students to solve problems such as the following using a written multiplication method: "134 cars were on each floor of the 6 storey car parking building. How many cars were there altogether?"

Students who are able to use a written form with understanding will need to proceed to Using Number Properties. If the students use an informal jotting or an expanded form, proceed with the lesson.

Equipment: Place value materials, such as place value blocks or beans, ice cream containers, and stickies (optional)

Using Materials

Tell the students that, before they model with materials, you are going to work out the answer to a simpler multiplication, 7 × 26, by recording it on paper. Use the vertical form and talk through the procedures as you carry them out.

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Twenty-six times seven.

$$\begin{array}{r} 26 \\ \times\ 7 \\ \hline \end{array}$$

Seven times six is 42.

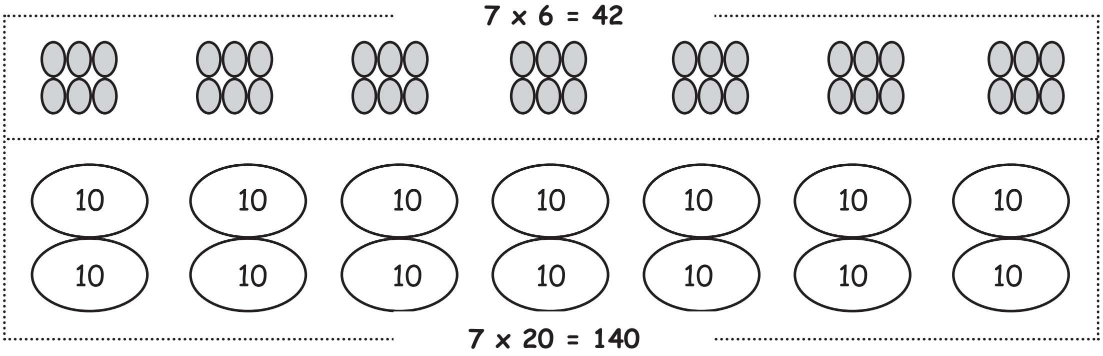
$$\begin{array}{r} {}^{4}26 \\ \times\ 7 \\ \hline 2 \end{array}$$

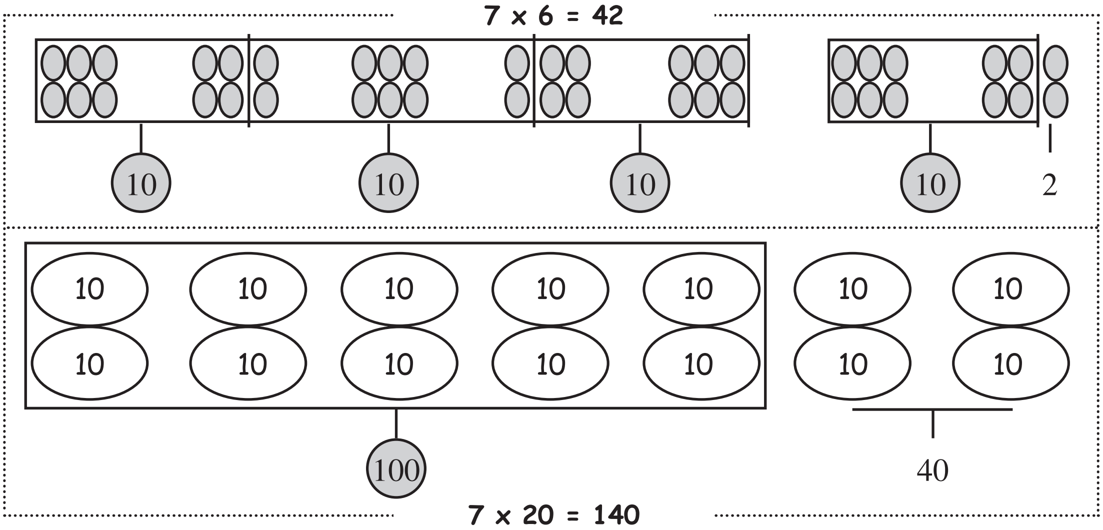
Seven times 20 is 140 plus 40 is 180.

$$\begin{array}{r} {}^{4}26 \\ \times\ 7 \\ \hline 182 \end{array}$$

Note: the 4 that is "carried" means 4 tens, or 40)

Provide the students with place value materials and ask them to explain how the written method works. Discuss what actions are performed on the materials to match the notation of the written form. Look to see that the students understand that place value partitioning is used to solve 7 × 26 = ☐.





Ones from the product of 7 × 6 = 42 are used to form four tens that can be added to the product of 7 × 20 = 140, so 7 × 26 = 182.

Ask the students to model other examples of the written form with equipment until they show an understanding of the process. Keep the number size low enough to enable easy modelling with the materials.

Examples might be:

37 × 4 = 148 23 × 8 = 184 69 × 3 = 207

Provide a similar experience for the division written form using the problem 92 ÷ 4.

"Fours into 90 are 20
with one 10 remaining."

$$\begin{array}{r} 2\phantom{\,^{1}2} \\ 4\,\overline{)9\,^{1}2} \end{array}$$

"Fours into 12 are three."

$$\begin{array}{r} 2\;\;3 \\ 4\,\overline{)9\,^{1}2} \end{array}$$

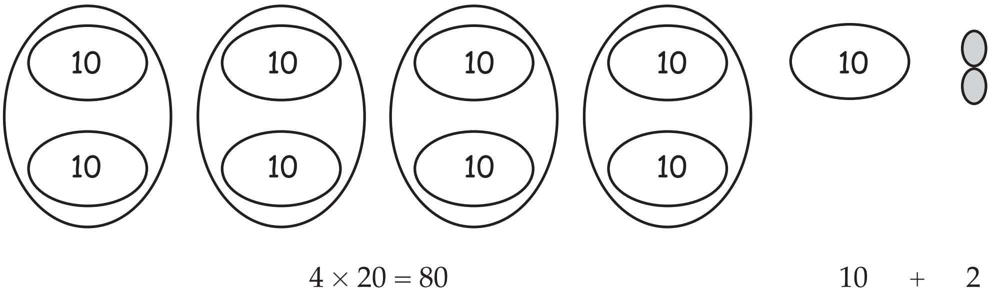
Provide the students with place value materials and ask them to explain what is occurring with the equipment at each stage of the written calculation.

Look for the understanding that one set of 10 is decomposed into 10 singles. The problem can be interpreted as both sharing (partitive division) or "how many sets of" (quotative division).

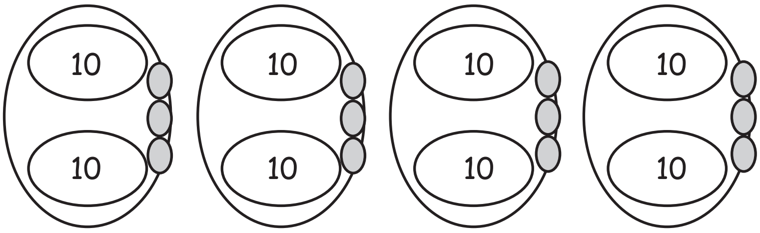
Twenty can be given to each share, leaving one 10 and two.

or

Twenty sets of four can be taken, leaving 10 and two remaining.

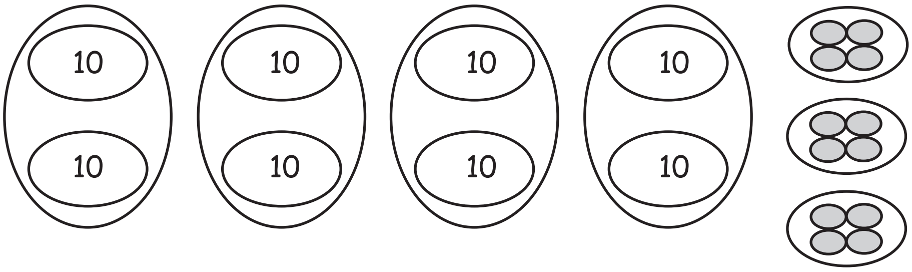


Ten and two can be decomposed to form 12 singles, which is shared into four sets of three.



or ...

Twelve can be used to form three more sets of four. The four twenties are the same as twenty fours; add the extra 3. This makes 23 fours.



Provide other division problems for the students to solve using the written form.

Use place value materials to illustrate what is happening to the numbers as the recording progresses.

Examples might be:

78 ÷ 3 = 26 85 ÷ 5 = 17 102 ÷ 6 = 17 198 ÷ 9 = 22

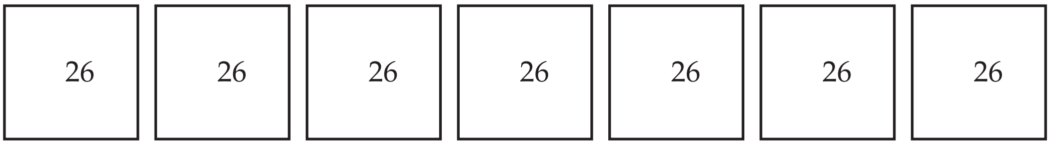
Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

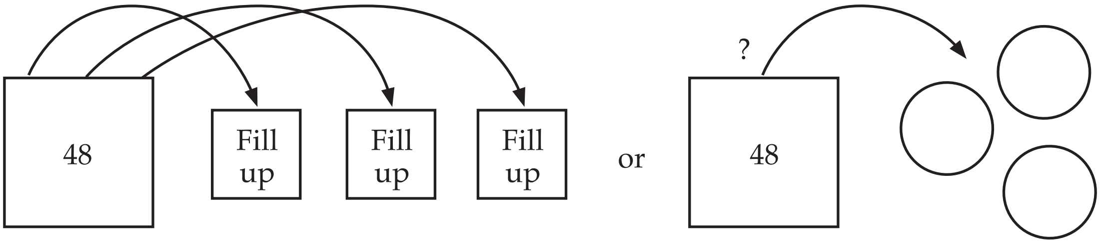
Shielding: Provide examples of multiplication and division problems for the students to solve using written forms.

Create the model of each problem but mask the materials with ice cream containers. Use stickies to label each container so that the students don't lose track of the numbers involved.

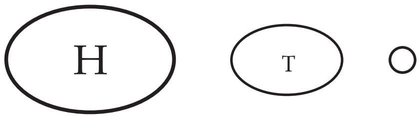
For example, model 7 × 26 as:



Model 48 ÷ 3 as:



Progress to drawing diagrams of the numbers, using symbols such as



to represent hundreds, tens, and ones respectively. This representation becomes more useful than manipulating the materials as number size increases.

Examples might be:

$\begin{array}{r} 36 \\ \times\ 4 \\ \hline \end{array}$	$\begin{array}{r} 47 \\ \times\ 3 \\ \hline \end{array}$	$\begin{array}{r} 29 \\ \times\ 5 \\ \hline \end{array}$	$\begin{array}{r} 34 \\ \times\ 6 \\ \hline \end{array}$	$\begin{array}{r} 17 \\ \times\ 9 \\ \hline \end{array}$	$\begin{array}{r} 23 \\ \times\ 7 \\ \hline \end{array}$
$4\overline{)96}$	$5\overline{)125}$	$3\overline{)81}$	$6\overline{)162}$	$8\overline{)256}$	$9\overline{)477}$

Using Number Properties

Increase the number size so that the students are required to generalise the actions involved in each written form.

Examples might be:

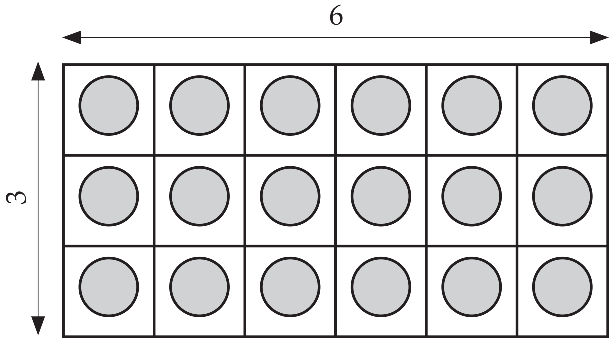
$\begin{array}{r} 547 \\ \times\ 5 \\ \hline \end{array}$	$\begin{array}{r} 609 \\ \times\ 3 \\ \hline \end{array}$	$\begin{array}{r} 374 \\ \times\ 4 \\ \hline \end{array}$	$\begin{array}{r} 875 \\ \times\ 6 \\ \hline \end{array}$	$\begin{array}{r} 1\,257 \\ \times\ 9 \\ \hline \end{array}$	$\begin{array}{r} 2\,503 \\ \times\ 7 \\ \hline \end{array}$
$5\overline{)390}$	$2\overline{)1\,748}$	$3\overline{)5\,310}$	$4\overline{)6\,060}$	$8\overline{)1\,856}$	$9\overline{)4\,770}$

Cross-Products

I am learning to multiply multi-digit whole numbers.

Key Mathematical Ideas

- Arrays are one context for multiplication. The factors in any multiplication can be represented by the number of objects or the length along each side and the product represented by the total number of objects or the area, e.g., for 3 × 6 = 18:

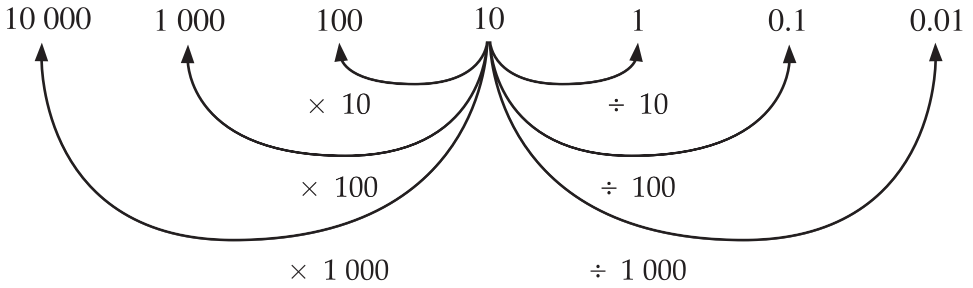


The large dotty array needs to be explored with simpler problems involving a single digit multiplier before moving to multi-digit numbers. It is helpful if the numbers are recorded on the edges of the rectangle.

- Place value is a powerful structure that makes complex counting problems relatively simple. Multiplicative ideas of place value include:
  - The place values are powers of ten:

Ten thousands	Thousands	Hundreds	Tens	Ones	Tenths	Hundreds
10 x 10 x 10 x 10	10 x 10 x 10	10 x 10	10	1	$\frac{1}{10}$	$\frac{1}{100}$
10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}

  - There is a "multiply by ten" relationship connecting *any place* with the place immediately to its left and a "divide by ten" relationship that connects it to the place immediately to its right:



  - Powers of ten multiply to give other powers of ten and divide to give powers of ten, e.g., 100 × 100 = 10 000 $(10^2 \times 10^2 = 10^4)$
    100 ÷ 1 000 = 0.1 $(10^2 \div 10^3 = 10^{-1})$
- Students are using an array model to support the development of multiplication strategies based on partitioning numbers, e.g., 23 × 37 as 20 × 30, 20 × 7, 3 × 30, and 3 × 7 (standard place value partition) or 20 × 37 and 3 × 37.
- Students can use a grid method to record their solutions. e.g., 23 × 37

x	20	3	
30	600	90	690
7	140	21	161
			851

- Students can be introduced to the standard algorithm for multiplication in its expanded form. This allows them to "chunk" the calculations into manageable pieces and lessens the load on working memory, e.g.:

$$\begin{array}{rl} 23 & \\ \text{x } 37 & \\ \hline 21 & (7 \times 3) \\ 140 & (7 \times 20) \\ 90 & (30 \times 3) \\ 600 & (30 \times 20) \\ \hline 851 & \end{array} \qquad \begin{array}{rl} 23 & \\ \text{x } 37 & \\ \hline 161 & (7 \times 23) \\ 690 & (30 \times 23) \\ \hline 851 & \end{array}$$

Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 × 10 and are able to reverse these facts to solve division problems, e.g., 9 × 8 = 72, so 72 ÷ 8 = 9.

Diagnostic Snapshot

Ask the students questions such as “18 × 33”.

Students who are able to use a written form with understanding will need to proceed to working with multiplication problems involving three-digit factors.

Equipment: Large dotty arrays (see Material Master 6–9), whiteboard pens (optional)

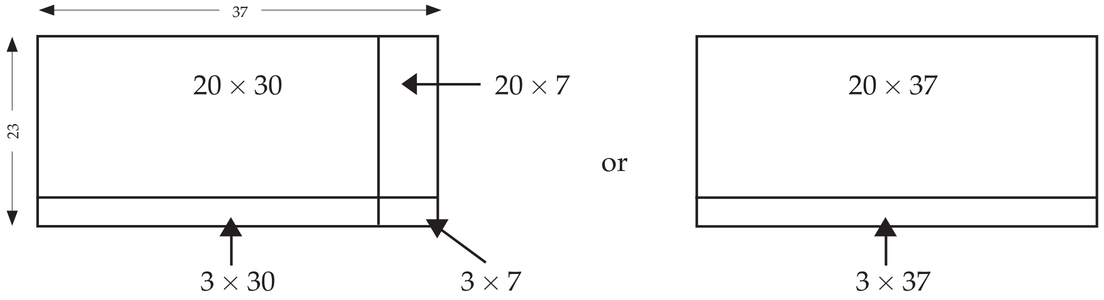
Using Materials

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Provide the students with a copy of the dotty array. If you are using laminated copies, whiteboard pens are very practical.

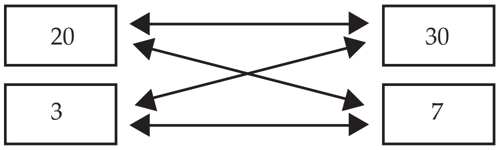
Write 23 × 37 on the board or modelling book and ask the students to draw a border around the array that represents it. Tell them to partition the array in any way they want that will make it easy to find the total number of dots.

Students are likely to use a variety of partitions including use of place value:



Discuss how the partitions relate to each other in that
3 × 37 = (3 × 30) + (3 × 7) and 20 × 37 = (20 × 30) + (20 × 7).

A key point is that the multiplication involves the idea of a cross-product. 23 × 37 can be calculated in this way, with each arrow representing a separate multiplication.



Note that students may suggest valuable strategies for 23 × 37 that are efficient but not transferable to more complex examples, for example,

23 × 37 = (25 × 37) – (2 × 37) = 925 – 74 = 851,

or 23 × 37 = (23 × 40) – (23 × 3) = 920 – 69 = 851.

Give the students other problems that can be modelled by partitioning the array. Good examples are:

42 × 15 = 630 26 × 24 = 624 18 × 33 = 594 45 × 35 = 1 575

Ensure the students have established a clear idea of the cross-product idea before proceeding to imaging.

Using Imaging and Number Properties

Provide further similar examples and ask the students to image the array and record the products that need to be calculated to ease memory load. Draw a border around the required array and hide it.

28 × 17 = 476　　19 × 43 = 817　　34 × 22 = 748　　40 × 34 = 1 360

Generalise the cross-product to include three-digit factors.

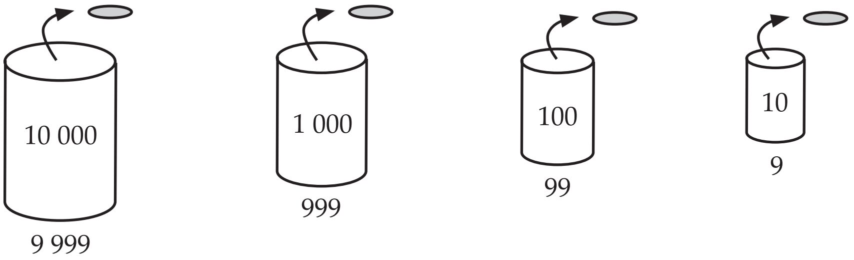
173 × 26 = (100 × 20) + (100 × 6) + (70 × 20) + (70 × 6) + (3 × 20) + (3 × 6)
= 2 000 + 600 + 1 400 + 420 + 60 + 18
= 4 498

Nines and Threes

I am learning to decide if a number is divisible by nine or three.

Key Mathematical Ideas

- Divisibility rules are quick "algorithms" to find out if a number will divide evenly into another, e.g., if a number is even, then it has either 0, 2, 4, 6, or 8 as its ones digit and will divide evenly by two: 3 576 divides evenly by 2, so 2 is a factor of 3 576 and 3 576 is a multiple of 2.
- Divisibility rules for 3 and 9 use place value partitioning. They work on the principle that taking one off each power of ten results in a number that is divisible by 3 or 9.



These are all divisible by 3 and 9.

- Other divisibility rules exist for 4, 5, 6, 7, and 8. These can easily be found by doing an Internet search on "divisibility rules".

Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 × 10 and are able to reverse these facts to solve division problems, e.g., 9 × 8 = 72 so 72 ÷ 8 = 9.

Diagnostic Snapshot

Ask the students questions such as:

"Is 5 688 divisible by 9?"

"Is 4 146 divisible by 3?"

Students who are able to use divisibility rules with understanding to answer these questions promptly can then explore other divisibility tests (*Book 8: Teaching Number Sense and Algebraic Thinking*, page 33).

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Equipment: Canisters of beans or other discrete place value materials, blank tens frames (Material Master 4–6), ice cream containers

Using Materials

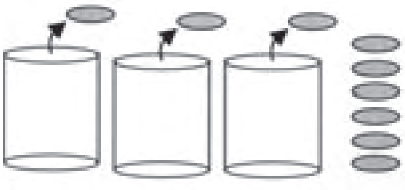
Ask the students to give you four different numbers that they know are in the nine times tables, for example, 27, 36, 54, 81. Make these numbers with place value material.

Tell the students that when a number is a multiple of nine, we say that it is "divisible by nine". Point out that the learning outcome they are looking for is a pattern that will help them tell if a number is divisible by nine without doing the division.

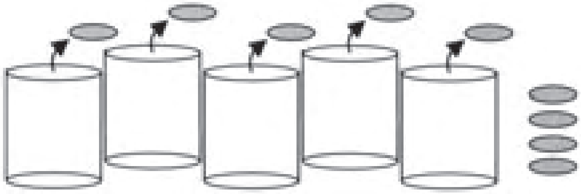
Ask, "How many nines do you get in 10?" (One set of nine with one left over.)

With the models of 36 and 54, etc., look at why these numbers are divisible by nine:

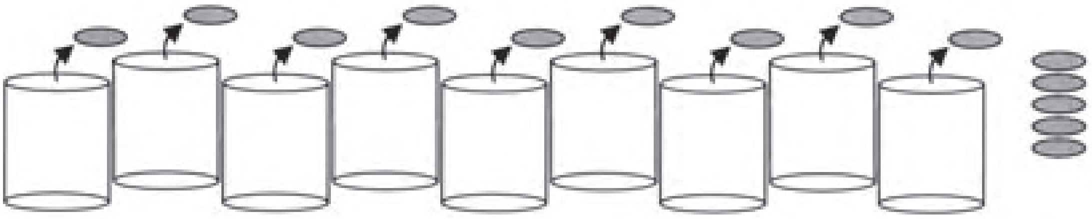
*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*



For 36, nine counters are left in each container and there are 3 + 6 = 9 extra ones. 36 is divisible by nine.



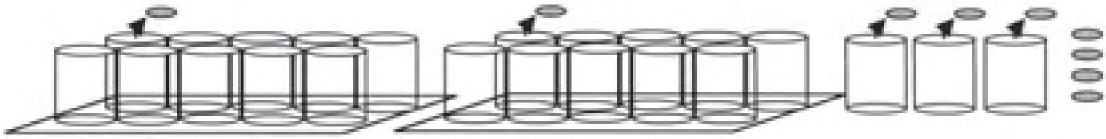
For 54, nine counters are left in each container and there are 5 + 4 = 9 extra ones. 54 is divisible by nine.



For 95, after putting nine counters in each container, the left over ones total 9 + 5 = 14 (sum of the digits in 95). 14 is not divisible by nine.

Extend this idea to three-digit numbers, like 234, that are divisible by nine. Ask, "How many nines do you get in 100?" (11 nines with one left over.)

Use blank tens frames with bean canisters placed on all 10 spaces to form hundreds. Make the numbers and find out whether or not they are divisible by nine.



For 234, 99 are left in every hundred, nine are left in every 10, and the extra ones total 2 + 3 + 4 = 9 (sum of the digits), so 234 is divisible by nine.

Model other three-digit numbers with materials, for example, 306, 576, 444. Find out if these numbers are divisible by nine. Look for the students to realise that the sum of the digits tells how many extra ones are available.

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The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Pose problems about divisibility by nine but mask the materials under plastic ice cream containers. Extend the students' thinking to include how many nines there must be in the number, and for numbers that are not divisible by nine, what the remainder after division by nine would be. Good examples are:

"Which of these numbers is divisible by nine?"

321 (3 + 2 + 1 = 6, so no), 279 (2 + 7 + 9 = 18, so yes), 198 (1 + 9 + 8 = 18, so yes), 426 (4 + 2 + 6 = 12, so no), 295 (2 + 9 + 5 = 16, so no), 997 (9 + 9 + 7 = 25, so no).

For each number ask, " How many nines are in each number?"

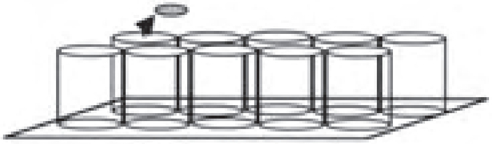
With 279, there are 2 × 11 = 22 nines in the hundreds, 7 nines in the tens, and 2 nines in the extra ones. That is a total of 22 + 7 + 2 = 31.

For each number that is not divisible by nine, ask, "What is the remainder when you divide [number] by nine?"

With 295, the extra ones total 2 + 9 + 5 = 16, 16 ÷ 9 = 1 remainder 7, so the remainder when 295 is divided by nine is seven.

Fold back to the materials if the students have difficulty imaging, particularly with the "How many nines?" and, "What is the remainder?" questions.

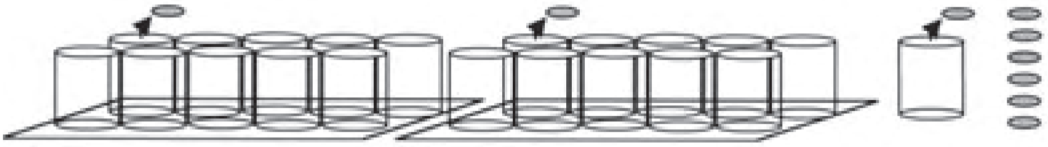
After sufficient examples, ask the students, "How could we use this same idea to find out if a number is divisible by three?" Some students may be able to generalise this idea.



There are 33 threes in every 100, with one left over.

There are three threes in every 10, with one left over.

To check if a number like 216 is divisible by three:



Taking one from each 100 and one from the 10 gives 2 + 1 + 6 = 9 extra ones.

Since nine is divisible by three, 216 is divisible by three.

Get the students to check other numbers for divisibility by three, folding back to the materials if necessary.

Using Number Properties

The students will show understanding of the number properties if they can solve the following problems mentally.

"For each number below decide:

Is it divisible by nine?

Is it divisible by three?

How many nines and threes are in the number?

What is the remainder when the number is divided by nine and three?"

852 1 008 666 2 074 5 688 6 251

Ask, "Can a number be divisible by nine but not be divisible by three?" (No) and

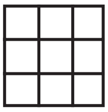
"Can a number be divisible by three but not be divisible by nine?" (Yes, e.g., 24)

Powerful Numbers

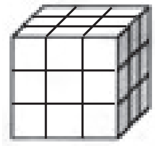
I am learning to solve multiplication problems with powers (exponents).

Key Mathematical Ideas

- Students are being introduced to the idea that powers (exponents) are an efficient way to express repeated products of the same number.
- Geometric models of squares and cubes are a good way to show numbers to the power of 2 and 3, e.g.,



$3^2 = 9$ (Three squared is nine.)



$3^3 = 27$ (Three cubed is twenty-seven.)

This acts as the foundation for students to work with powers beyond cubes using number properties.

- Students' typical misconceptions relating to powers of zero and one, e.g., 2^o and 2^1, are addressed through patterning.

$$\begin{tikzcd}
4^3 & 4^2 & 4^1 & 4^0 & 4^{-1} \\
4 \times 4 \times 4 & 4 \times 4 & 4 & & \\
64 \arrow[r, bend right, "\div 4"'] & 16 \arrow[r, bend right, "\div 4"'] & 4 \arrow[r, bend right, "\div 4"'] & 1 \arrow[r, bend right, "\div 4"'] & \frac{1}{4}
\end{tikzcd}$$

Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 × 10 and are able to reverse these facts to solve division problems, e.g., $9 \times 8 = 72$, so $72 \div 8 = 9$.

Diagnostic Snapshot 📷

Ask the students questions such as:

"Solve 2^6 mentally and record the expanded notation."

"If $3^3 = 27$ and $3^4 = 81$, what power of 3 will the answer to 27×81 be?" (seven because $3^3 \times 3^4 = 3^7$)

If $4^5 = 1\,024$ and $4^3 = 64$, solve $1\,024 \div 64 = \square$ using exponents. (16 because $4^2 \times 4^3 = 4^5$, so $16 \times 64 = 1\,024$)

Students who are able to solve the problems with understanding will need to proceed to Using Number Properties.

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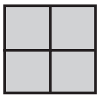
Equipment: Unilink cubes, rubber bands, calculators

Using Materials

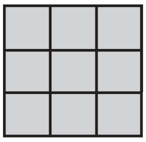
Get the students to work together to make four stacks of two cubes, nine stacks of three cubes, 16 stacks of four cubes, and 25 stacks of five cubes.

Put four loose cubes down and form them into a 2×2 square shape. Do the same with nine cubes (3×3), 16 cubes (4×4), and 25 cubes (5×5).

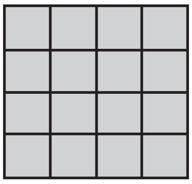
Ask the students to work out the total number of cubes in each square. Tell the students that these numbers are sometimes referred to as “two squared”, “three squared,” “four squared”, etc. They are written as 2^2, 3^2, 4^2, etc.



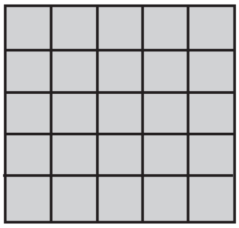
$2^2 = 4$



$3^2 = 9$



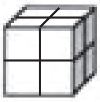
$4^2 = 16$



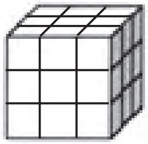
$5^2 = 25$

Tell the students to put one of the product numbers (e.g., 16) into the calculator and press the square root button ($\sqrt{\ \ }$). Repeat for the other three products. Ask the students what the square root button does. (It gives the length of each side of the square, e.g. $\sqrt{16} = 4$.) Test this out by getting the students to calculate $\sqrt{100} = 10$. A 10×10 square has an area of 100.

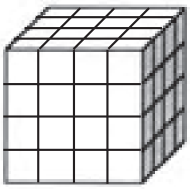
Using rubber bands, make the stacks of cubes into $2 \times 2 \times 2$, $3 \times 3 \times 3$, $4 \times 4 \times 4$, and $5 \times 5 \times 5$ cubes. Ask the students to calculate how many small cubes make up each larger cube. Encourage them to use mental strategies, such as $4 \times 4 = 16$, $4 \times 16 = 8 \times 8 = 64$, that involve splitting factors. Tell the students that these numbers are sometimes called “two cubed”, “three cubed,” “four cubed,” etc., and are written as 2^3, 3^3, 4^3, etc.



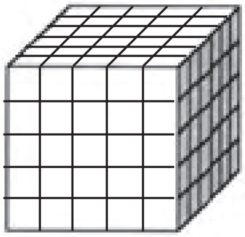
$2^3 = 8$



$3^3 = 27$



$4^3 = 64$



$5^3 = 125$

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Ask the students to describe what models for these powers would look like and what the number would be:

8^2 (64) 6^3 (216) $\sqrt{81}$ (9) 100^2 (10 000) 10^3 (1 000) $\sqrt{400}$ (20)

Tell the students that numbers can be raised to the powers of 4, 5, 6, and so on. The number involved becomes very large, so it is not usually practical to make models of them. For example, 5^4 (5 to the power of 4) $= 5 \times 5 \times 5 \times 5 = 625$. Get the students to calculate numbers to higher powers, such as $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Using Number Properties

Draw up a table of the power of two (exponents with two as a base, as in the first row of the table below). Leave the values of 2^0 and 2^1 empty (in the second row). Fill in the table as the students mentally calculate the next number each time by doubling:

Number	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Value			4	8	16	32	64	128	256	512	1 024

Ask the students what 2^0 and 2^1 must be if the pattern of doubling to the right and halving to the left still holds ($2^1 = 2$ and $2^0 = 1$).

Tell the students to work out $8 \times 32 = 256$ mentally. Note that the answer is in the table as well. Record this as $2^3 \times 2^5 = 2^8$. Give the students two other similar problems, e.g., $64 \times 8 = 512$ ($2^6 \times 2^3 = 2^9$) and $256 \times 4 = 2^8 \times 2^2 = 2^{10}$.

Some students may note that adding the exponents of the factors gives the exponent of the product. This idea can be tested out with other examples for the powers of two table and also by constructing a powers of three table:

Number	3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8	3^9	3^{10}
Value	1	3	9	27	81	243	729	2 187	6 561	19 683	59 049

Challenge the students to explain why $3^2 \times 3^3 = 3^5$.

Record this as: $(3 \times 3) \times (3 \times 3 \times 3) = (3 \times 3 \times 3 \times 3 \times 3)$. The students should note that the factors are the same, they are just grouped in different ways.

Provide other examples for them to explain, e.g., $2^4 \times 2^5 = 2^9$.

Challenge the students to use their calculators to find out what happens when powers are divided. For example: $2^{10} \div 2^4 = 2^6$, $2^7 \div 2^3 = 2^4$, $3^5 \div 3^4 = 3^1$, $3^9 \div 3^5 = 3^4$.

The students should notice that the exponents are subtracted to find the exponent of the quotient. They may like to investigate what a number to the power of one-half must be ($16^{\frac{1}{2}} = \sqrt{16} = 4$ and $81^{\frac{1}{2}} = \sqrt{81} = 9$).

Learning Experiences to Move Students from Advanced Multiplicative–Early Proportional Part-Whole to Advanced Proportional Part-Whole

Knowledge to be Developed

As students' understanding of multiplication and division progresses at this stage, it is essential that the following knowledge is developing simultaneously:

- Common Factors; highest common factor: *Book 8: Number Sense and Algebraic Thinking*, page 39
- Least common multiple to 10: *Book 8: Number Sense and Algebraic Thinking*, page 40
- Divisibility rules for 4, 6, 8: *Book 8: Number Sense and Algebraic Thinking*, page 33, with Material Master 8–24

Little Bites at Big Multiplications and Divisions

I am learning to solve multiplication and division problems using factors of factors.

Key Mathematical Ideas

Students are learning to solve multi-digit multiplication and division problems by using factors of factors, e.g., 216 ÷ 12 as 216 ÷ 2 ÷ 2 ÷ 3, i.e., 216 ÷ 2 = 108, 108 ÷ 2 = 54, 54 ÷ 3 = 18 (the factors of 12 are 2 × 2 × 3). Usually, this idea is only practical if the factors are small prime numbers such as 2, 3, and 5.

Key Mathematical Knowledge

Check that the students have quick recall of multiplication and division facts up to 10 × 10 and are able to reverse these facts to solve division problems, e.g., 9 × 8 = 72, so 72 ÷ 8 = 9. Students also need to know factors of numbers up to 100, e.g., for 81: 1, 3, 9, 27.

Diagnostic Snapshot 📷

Ask the students questions such as "Solve 1 536 ÷ 24."

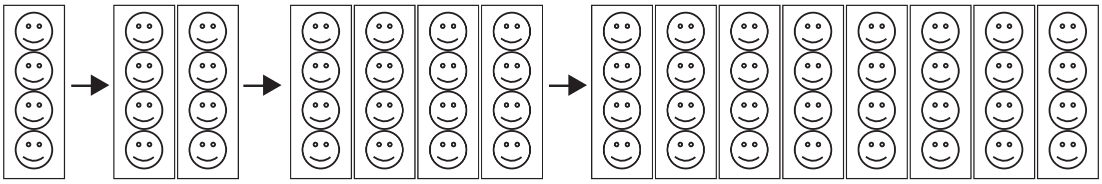
Share strategies and determine whether any students used the factors of factors method as one possible method, i.e., 1 536 ÷ 3 = 512, 512 ÷ 2 = 256, 256 ÷ 2 = 128, 128 ÷ 2 = 64.

Students who are able to solve the problems using factors of factors will need to proceed to Using Number Properties.

Equipment: Animal strips

Using Materials

Give out sets of animal strips to the students. Tell them to take a collection of the same strips, e.g., take all the four strips. Follow this sequence of instructions for laying down cards and recording:



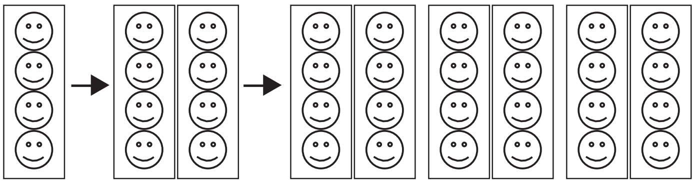
Lay down one card ... double the number and record $2 \times \square$ ... double this amount and record $2 \times 2 \times \square$ ... double the amount again.

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Ask the students how they should record what they have. They may give you responses about their personal collections, for example, "I've got eight sixes", that can in turn be generalised as $2 \times 2 \times 2 \times \square$ is the same as $8 \times \square$.

Ask them how many times their number they will have if the eight sets are doubled (sixteen sets). Record this as $2 \times 2 \times 2 \times 2 \times \square$ is the same as $16 \times \square$. Focus on what connection there is between the $2 \times 2 \times 2 \times 2$ and the 16 ($2 \times 2 \times 2 \times 2$ is a factorisation of 16).

Tell the students to use the same collection of strips to work out what happens when a number is multiplied by two, then multiplied by three. Record their ideas as expressions, for example, $3 \times (2 \times \square)$:



$2 \times \square$ $3 \times 2 \times \square$ (is the same as $6 \times \square$)

Provide other examples for students to explore such as:

$2 \times (5 \times \square)$ (is the same as $10 \times \square$)

$5 \times (3 \times \square)$ (is the same as $15 \times \square$)

Explore whether the same idea can be applied to division. Begin with a set-up array composed of animal strips, for example 8 sets of the same strip. Ask students to record a division equation for their array, for example $32 \div 8 = 4$. Discuss what feature of the array gives them the answer in this case (the number on each strip).

Ask them to halve the number of strips and record the division equation, for example, $32 \div 8 = 16 \div 4$. Ask them to halve their array again and record the equation. This can be generalised as $32 \div 2 \div 2 \div 2$ is the same as $32 \div 8$. Reinforce the significance of $2 \times 2 \times 2$ being a factorisation of 8.

Provide similar division examples, such as $24 \div 2 \div 3$ is the same as $24 \div 6$, and $40 \div 2 \div 5$ is the same as $40 \div 10$.

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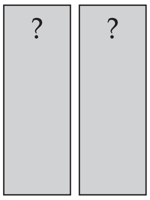
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

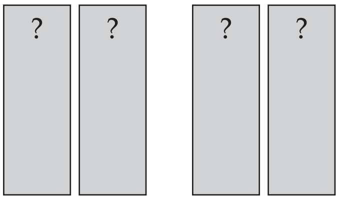
Using Imaging

Ask the students to image repeated multiplication and division examples and to represent them by recording box diagrams. Examples might be:

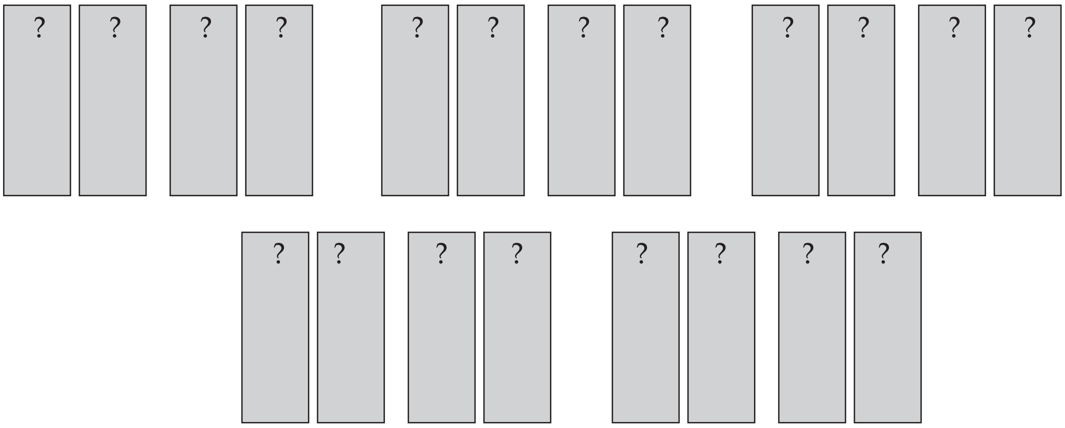
2 × □



2 × 2 × □

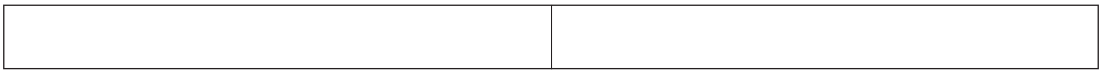


5 × 2 × 2 × □ ... is the same as 20 × □

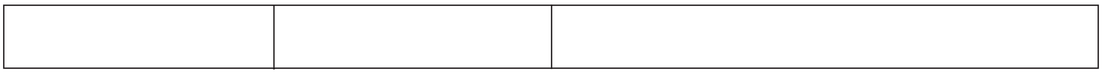


or 2 × 3 × 3 × □ is the same as 18 × □
and 3 × 3 × 3 × □ is the same as 27 × □.

Provide division examples like the steps below.



□ ÷ 2



□ ÷ 2 ÷ 2

□ ÷ 2 ÷ 2 ÷ 3 is the same as ÷ 12

or □ ÷ 3 ÷ 5 is the same as □ ÷ 15
and □ ÷ 2 ÷ 2 ÷ 3 ÷ 3 is the same as □ ÷ 36.

Using Number Properties

Provide examples for students to solve using repeated division by factors. Good examples might be:

8×43 as $43 \times 2 = 86$, $86 \times 2 = 172$, $172 \times 2 = 344$

12×56 as $56 \times 2 = 112$, $112 \times 2 = 224$, $224 \times 3 = 672$

15×68 as $68 \times 5 = 340$, $340 \times 3 = 1\,020$

18×26 as $26 \times 2 = 52$, $52 \times 3 = 156$, $156 \times 3 = 468$

36×77 as $77 \times 2 = 154$, $154 \times 2 = 308$, $308 \times 3 = 924$, $924 \times 3 = 2\,772$

$216 \div 12$ as $216 \div 2 = 108$, $108 \div 2 = 54$, $54 \div 3 = 18$

$688 \div 16$ as $688 \div 2 = 344$, $344 \div 2 = 172$, $172 \div 2 = 86$, $86 \div 2 = 43$

$612 \div 18$ as $612 \div 2 = 306$, $306 \div 3 = 102$, $102 \div 3 = 34$

$1\,536 \div 24$ as $1\,536 \div 3 = 512$, $512 \div 2 = 256$, $256 \div 2 = 128$, $128 \div 2 = 64$

Practice

Students will enjoy playing the game of *Factor Leapfrog* (see Material Master 6–10) to consolidate these strategies for multiplication.

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